Time Series Econometrics
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Homework 2: Solutions

Homework 2: Question 1

(a) Give the state-space representation of the following models:

i. Unobserved component model with a stochastic drift

$$\begin{aligned} y_t &= \tau_t + c_t, \\ \tau_t &= g_{t-1} + \tau_{t-1} + v_t, \qquad v_t \sim iid\, N(0, \sigma_v^2) \\ g_t &= g_{t-1} + w_t, \qquad w_t \sim iid\, N(0, \sigma_w^2) \\ c_t &= \phi_1 c_{t-1} + \phi_2 c_{t-2} + e_t, \qquad e_t \sim iid\, N(0, \sigma_e^2). \end{aligned}$$

Note that in this model, the stochastic trend, τ_t, has a drift term (g_t) modeled as a random walk.

ii. Time-varying parameter model

$$\begin{aligned} y_t &= x_{1,t}\beta_{1,t} + x_{2,t}\beta_{2,t} + u_t, \quad u_t \sim iid\, N(0, \sigma^2), \\ \beta_{i,t} &= \beta_{i,t-1} + v_{i,t}, \quad v_{i,t} \sim iid\, N(0, \sigma_i^2). \end{aligned}$$

(b) Briefly discuss how you would estimate a model given in a state space form using maximum likelihood.

Answer:

(a) State-space representation is of form Transition equation

$$\beta_t = F\beta_{t-1} + v_t, \qquad v_t \sim N(0, Q)$$

Observation equation:

$$y_t = H_t\beta_t + e_t, \qquad e_t \sim N(0, R)$$

i. Unobserved component model with a stochastic drift

$$\begin{aligned} y_t &= \tau_t + c_t, \\ \tau_t &= g_{t-1} + \tau_{t-1} + v_t, \qquad v_t \sim iid\, N(0, \sigma_v^2) \\ g_t &= g_{t-1} + w_t, \qquad w_t \sim iid\, N(0, \sigma_w^2) \\ c_t &= \phi_1 c_{t-1} + \phi_2 c_{t-2} + e_t, \qquad e_t \sim iid\, N(0, \sigma_e^2). \end{aligned}$$

can be in terms of:
Transition equation

$$\begin{bmatrix} \tau_t \\ g_t \\ c_t \\ c_{t-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \phi_1 & \phi_2 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \tau_{t-1} \\ g_{t-1} \\ c_{t-1} \\ c_{t-2} \end{bmatrix} + \begin{bmatrix} v_t \\ w_t \\ e_t \\ 0_t \end{bmatrix}$$

Observation equation:

$$y_t = \begin{bmatrix} 1 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \tau_t \\ g_t \\ c_t \\ c_{t-1} \end{bmatrix}$$

ii. Time-varying parameter model

$$\begin{aligned} y_t &= x_{1,t}\beta_{1,t} + x_{2,t}\beta_{2,t} + u_t, \quad u_t \sim iid\, N(0, \sigma^2), \\ \beta_{i,t} &= \beta_{i,t-1} + v_{i,t}, \quad v_{i,t} \sim iid\, N(0, \sigma_i^2). \end{aligned}$$

can be in terms of
Transition equation:

$$\begin{bmatrix} \beta_{1,t} \\ \beta_{2,t} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \beta_{1,t-1} \\ \beta_{2,t-1} \end{bmatrix} + \begin{bmatrix} v_{1,t} \\ v_{2,t} \end{bmatrix}$$

Observation equation:

$$y_t = \begin{bmatrix} x_{1,t} & x_{2,t} \end{bmatrix} \begin{bmatrix} \beta_{1,t} \\ \beta_{2,t} \end{bmatrix} + u_t$$

(b) Once the model is in the state-space form we can use Kalman-Filter to construct a sequence of forecast errors and, using the prediction error decomposition, construct a likelihood function. Given that the PED and likelihood function are depends on the model parameters in matrices F, H, μ, Q, R, we can use numerical maximization of likelihood to obtain numerical estimate of the parameters of interest.

Homework 2: Question 2: Sims (1980)
Stata Exercise