

# Lecture: State-Space Form

222061-1617: Time Series Econometrics

Spring 2023

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# Outline

Outline:

- 1 Prediction Error Decomposition
- 2 State-Space Form
- 3 Kalman Filter

# PREDICTION ERROR DECOMPOSITION

# Likelihood

- Consider a covariance-stationary  $\{Y_t\}$

$$\begin{bmatrix} y_1 \\ \vdots \\ y_T \end{bmatrix} = \tilde{y}_T \sim N(\mu_{T \times 1}, \Omega_{T \times T}),$$

- Since it is covariance stationary process, each  $Y_t$  has the same mean and variance,  $\omega_{11} = \sigma^2 = \omega_{22} = \omega_{TT}$ .

$$\Omega = \begin{bmatrix} \omega_{11} & \omega_{12} & & \omega_{1T} \\ \omega_{21} & \omega_{22} & & \vdots \\ \vdots & & \ddots & \\ \omega_{T1} & \dots & & \omega_{TT} \end{bmatrix} = \begin{bmatrix} \gamma_0 & \gamma_1 & \dots & \gamma_{T-1} \\ \gamma_1 & \gamma_0 & & \vdots \\ \vdots & & \ddots & \\ \gamma_{T-1} & \dots & & \gamma_0 \end{bmatrix}, \text{ as } \begin{cases} \omega_{jj} = \gamma_0, \\ \omega_{ij} = \gamma_{i-j}, \\ i > j. \end{cases}$$

- The likelihood function:

$$L(\tilde{\theta} | \tilde{y}_T) = (2\pi)^{-\frac{T}{2}} \det(\Omega)^{-\frac{1}{2}} e^{-\frac{1}{2}(\tilde{y}_T - \mu)' \Omega^{-1} (\tilde{y}_T - \mu)}.$$

# Factorization

- For large  $T$ ,  $\Omega$  might be large and difficult to invert.
- Since  $\Omega$  is positive definite symmetric matrix, there exists a unique, triangular factorization of  $\Omega$ ,

$$\Omega = AfA',$$

where

$$f_{T \times T} = \begin{bmatrix} f_1 & 0 & \cdots & 0 \\ 0 & f_2 & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & & f_T \end{bmatrix}, \quad f_t > 0 \forall t \quad \text{diagonal matrix}$$

$$A_{T \times T} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ a_{21} & 1 & & \vdots \\ \vdots & & \ddots & \\ a_{T1} & a_{T2} & \cdots & 1 \end{bmatrix}, \quad \text{lower unit triangular matrix}$$

# Prediction errors

- Define  $\eta = A^{-1}(\tilde{y}_T - \mu)$ ,

$$E(\eta)$$

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- Since  $f$  is diagonal,  $\eta$  is a series of random variables that are uncorrelated with each other, i.e.  $E(\eta_t \eta_\tau) = 0$ ,  $\forall t \neq \tau$ .
- Since  $A\eta = (\tilde{y}_T - \mu)$  and  $A$  is a lower-triangular matrix with 1s along the principal diagonal,

$$\eta_1 = y_1 - \mu$$

$$\eta_2 = y_2 - \mu - a_{11}^* \eta_1$$

$$\eta_3 = y_3 - \mu - a_{21}^* \eta_1 - a_{22}^* \eta_2$$

$$\vdots$$

$$\eta_T = y_T - \mu - \sum_{i=1}^{T-1} a_{T-1,i}^* \eta_i$$

where  $a_{i,j}^* = A_{i+1,j}$

# Prediction errors

- Since  $E(\eta_t \eta_\tau) = 0$ ,

$$E(\eta_2 \eta_1) = E((y_2 - \mu - a_{11}^*(y_1 - \mu))(y_1 - \mu)) = 0$$

so the  $a_{11}^*$  is the coefficient of the linear projection of  $(y_2 - \mu)$  on  $(y_1 - \mu)$  and  $\eta_2 = y_2 - y_{2|1}$  is a prediction error .

- Similarly,  $E(\eta_3 \eta_2) = E(\eta_3 \eta_1) = 0$  imply that  $a_{21}^*$  and  $a_{22}^*$  are linear projection coefficients of  $(y_3 - \mu)$  on  $(y_2 - \mu)$  and  $(y_1 - \mu)$ , with  $\eta_3 = y_3 - y_{3|2}$
- Therefore,  $\eta_t$  is  $t^{\text{th}}$  element of  $\eta_{T \times 1} =$  prediction error  $y_t - \hat{y}_{t|t-1}$ .

# Likelihood

- The likelihood function can be rewritten as:

$$L(\tilde{\theta}|\tilde{y}_T) = (2\pi)^{-\frac{T}{2}} \det(AfA')^{-\frac{1}{2}} e^{-\frac{1}{2}(\tilde{y}_T - \mu)'(AfA')^{-1}(\tilde{y}_T - \mu)}$$

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- $A$  is lower triangular with 1s along the principal diagonal so  $\det(A) = 1$  and

$$\det(AfA) = \det(A) \cdot \det(f) \cdot \det(A') = \det(f).$$

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- Then,

$$\begin{aligned} L(\tilde{\theta}|\tilde{y}_T) &= (2\pi)^{-\frac{T}{2}} \det(f^{-1})^{-\frac{1}{2}} e^{-\frac{1}{2}\eta'(f^{-1})^{-1}\eta} \\ &= \prod_{t=1}^T \left( \frac{1}{\sqrt{2\pi f_t}} e^{-\frac{1}{2} \frac{\eta_t^2}{f_t}} \right), \end{aligned}$$

where  $\eta_t$  is  $t^{\text{th}}$  element of  $\eta_{T \times 1} = \text{prediction error } y_t - \hat{y}_{t|t-1}$ ,

$$\hat{y}_{t|t-1} = \sum_{i=1}^{t-1} a_{t,i}^* y_i, \quad i = 2, 3, \dots, T.$$

# Kalman Filter

Note: Given  $y_t \sim N(\mu, \Omega)$ ,

$$\eta_t | \Omega_{t-1} \sim N(0, f_t),$$

where  $f_t$  is an  $(t, t)$  diagonal element of  $f$  matrix,

$$\ln L = -\frac{1}{2} \sum_{t=1}^T \ln(2\pi f_t) - \frac{1}{2} \sum_{t=1}^T \frac{\eta_t^2}{f_t},$$

since  $\eta_t \sim N$  and independent of each other.

- The Kalman filter recursively calculates linear projection of  $y_t$  on past information  $\Omega_{t-1}$  for any model that can be cast in state-space form.
- *Kalman filter: for any structure it solves for linear prediction.*

# STATE-SPACE



# Measurement (Observation) Equation

General form that encompasses a wide variety of models.

## 1 Measurement (Observation) Equation

- Represent the static relationship between observed variables (data) and unobserved state variables.

$$y_t = H_t \beta_t + A z_t + e_t,$$

where  $y_t$  denotes observed data,  $\beta_t$  is a state vector that captures the dynamics,  $z_t$  is exogenous, observed variables *for example, lagged values of  $y_t$  but also other data*, and  $e_t$  is an error term,

$$e_t \sim N(0, R).$$

The existence of the state vector makes this representation not a simple linear model.

# Transition (State) Equation

## ② Transition (State) Equation

- *Captures the dynamics in the system, causes the system to go on and on.*

$$\beta_t = \tilde{\mu} + F\beta_{t-1} + v_t,$$

where  $\tilde{\mu}$  is a vector of constants,  $F$  is the transition matrix, and  $v_t$  is an error vector,

$$v_t \sim N(0, Q).$$

- *Like AR(1) but in vector/matrix form.*

# Transition (State) Equation

$$\beta_t = \tilde{\mu} + F\beta_{t-1} + v_t,$$

- The state vector has an AR(1) kind of representation.
- Describes evolution of state vector.
- These state vectors can be unobservable.
- Transition equation can be used to get information about the unobservable, conditioning on data which is observable (Bayesian).

# Error terms

Error terms:

$$e_t \sim N(0, R), \quad v_t \sim N(0, Q),$$

where  $R, Q$  are var-cov matrices and

$$E[e_t v_\tau'] = 0, \quad \forall t, \tau$$

- Restrictive assumption
- The model can be represented in a way that is not very restrictive.
- Even with  $E[e_t v_\tau'] \neq 0$  we can estimate the model with (modified) Kalman Filter but it becomes more complicated.
- The normality assumption might not be always good...  
...but it allows to use MLE.

# Examples: AR(2)

Consider an AR(2) process

$$\begin{aligned}y_t &= c + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \varepsilon_t, \\ \varepsilon_t &= WN(0, \sigma^2).\end{aligned}$$

State equation

Observation equation

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State equation

$$\begin{bmatrix} y_t \\ y_{t-1} \end{bmatrix} = \begin{bmatrix} \phi_1 & \phi_2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} y_{t-1} \\ y_{t-2} \end{bmatrix} + \begin{bmatrix} c \\ 0 \end{bmatrix} + \begin{bmatrix} \varepsilon_t \\ 0 \end{bmatrix}$$

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Observation equation

$$y_t = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} y_t \\ y_{t-1} \end{bmatrix}.$$

# Examples: AR(2) again

Consider again an AR(2) process

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State equation

$$\beta_t = \begin{bmatrix} y_t \\ \phi_2 y_{t-1} \end{bmatrix} = \begin{bmatrix} \phi_1 & 1 \\ \phi_2 & 0 \end{bmatrix} \begin{bmatrix} y_{t-1} \\ \phi_2 y_{t-2} \end{bmatrix} + \begin{bmatrix} c \\ 0 \end{bmatrix} + \begin{bmatrix} \varepsilon_t \\ 0 \end{bmatrix}$$

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Observation equation

$$y_t = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} y_t \\ \phi_2 y_{t-1} \end{bmatrix}.$$

# Examples: MA(1)

- Consider an MA(1) process

$$y_t = \mu + \varepsilon_t + \theta\varepsilon_{t-1},$$

$$\varepsilon_t = WN(0, \sigma^2).$$

State equation

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- Then

State equation

$$\beta_t = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \beta_{t-1} + \begin{bmatrix} 1 \\ \theta \end{bmatrix} \varepsilon_t$$

Observation equation

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State equation

$$\beta_t = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \beta_{t-1} + \begin{bmatrix} 1 \\ \theta \end{bmatrix} \varepsilon_t$$

Observation equation

$$y_t = \mu + [1 \quad 0] \beta_t.$$

# Examples: ARMA(1,1)

ARMA(1,1):

Set  $\mu = 0$ ,

$$y_t = \phi y_{t-1} + \varepsilon_t + \theta \varepsilon_{t-1}, \quad \varepsilon_t \sim N(0, \sigma^2).$$

- *There might be more than one way to represent a model in a state-space form.*
- *There might be differences in efficiency between different ways.*

# Examples: ARMA(1,1)

$$y_t = \phi y_{t-1} + \varepsilon_t + \theta \varepsilon_{t-1}, \quad \varepsilon_t \sim N(0, \sigma^2).$$

## State equation:

- The general form

$$\beta_t = F\beta_{t-1} + v_t.$$



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- Putting  $y_t = \phi y_{t-1} + \theta \varepsilon_{t-1} + \varepsilon_t$  in a matrix notation:

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$$\begin{bmatrix} y_t \\ \varepsilon_t \\ \beta_t \end{bmatrix} = \begin{bmatrix} \phi & \theta \\ \mathbf{0} & \mathbf{0} \\ F \end{bmatrix} \begin{bmatrix} y_{t-1} \\ \varepsilon_{t-1} \\ \beta_{t-1} \end{bmatrix} + \begin{bmatrix} \varepsilon_t \\ \varepsilon_t \\ v_t \end{bmatrix},$$

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$$\begin{bmatrix} y_t \\ \varepsilon_t \\ \beta_t \end{bmatrix} = \begin{bmatrix} \phi & \theta \\ 0 & 0 \\ F \end{bmatrix} \begin{bmatrix} y_{t-1} \\ \varepsilon_{t-1} \\ \beta_{t-1} \end{bmatrix} + \begin{bmatrix} \varepsilon_t \\ \varepsilon_t \\ v_t \end{bmatrix},$$

and  $v_t \sim N(0, Q)$ ,  $Q = \begin{bmatrix} \sigma^2 & \sigma^2 \\ \sigma^2 & \sigma^2 \end{bmatrix}$ .

$y_t$  – observable,  $\varepsilon_t$  – unobservable, forecast error

# Examples: ARMA(1,1)

Observation equations:

$$y_t = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} y_t \\ \varepsilon_t \\ \beta_t \end{bmatrix}$$
$$y_t \qquad \qquad H$$

no exogenous variables:  $A = 0$ , also  $R = 0$ .

- $y_t = H\beta_t$

The parameters  $\phi, \theta, \sigma^2$  are captured in  $F, Q$  matrices. The Kalman Filter will estimate them.

- *For KF what goes in  $\beta_t$  doesn't matter.*
- *Only parameters  $F, Q, H, R$  will matter.*
- *The state vector is now defined by  $F, Q, H$ , and the observations.*

# ARMA(1,1): Alternative Representation

*A more “elegant” ( i.e. easier for computation) representation.*

Lag notation (alternative representation for ARMA(1,1))

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Define  $x_t = (1 - \phi L)^{-1}\varepsilon_t$

$$(1 - \phi L)x_t = \varepsilon_t, \quad (x_t \text{ is AR}(1), \text{ not observed})$$

$$x_t - \phi x_{t-1} = \varepsilon_t$$

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$$\begin{aligned}(1 - \phi L)x_t &= \varepsilon_t, & (x_t \text{ is AR}(1), \text{ not observed}) \\x_t - \phi x_{t-1} &= \varepsilon_t\end{aligned}$$

Then,

$$\begin{aligned}y_t &= (1 + \theta L)x_t \\y_t &= x_t + \theta x_{t-1}.\end{aligned}$$

So  $y_t$  is a linear combination of 2 unobservable AR(1) processes,  $x_t$  and  $x_{t-1}$ .

# ARMA(1,1): State-Space

Observation equation (*all randomness in the state equation*)

$$y_t = H\beta_t,$$

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Observation equation (*all randomness in the state equation*)

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where

$$y_t = \begin{bmatrix} 1 & \theta \end{bmatrix} \begin{bmatrix} x_t \\ x_{t-1} \end{bmatrix}$$

- *Inside  $H$  there are parameters to be estimated.*
- *$A = 0$ , no exogenous,  $R = 0$  as the observable equation is just the identity (no randomness of  $e_t$ ).*

# ARMA(1,1): State-Space

## State equation

$$\begin{bmatrix} x_t \\ x_{t-1} \\ \beta_t \end{bmatrix} = \begin{bmatrix} \phi & 0 \\ 1 & 0 \\ & F \end{bmatrix} \begin{bmatrix} x_{t-1} \\ x_{t-2} \\ \beta_{t-1} \end{bmatrix} + \begin{bmatrix} \varepsilon_t \\ 0 \\ v_t \end{bmatrix},$$

so

$$v_t \sim N(0, Q), \quad Q = \begin{bmatrix} \sigma^2 & 0 \\ 0 & 0 \end{bmatrix}.$$

- So  $\phi$  is in  $F$ ,  $\theta$  in  $H$ , and  $\sigma^2$  in  $Q$ .

Given  $F, Q, H, A, R$  and data ( $y_t$ 's), use Kalman Filter to find prediction error decomposition of joint likelihood for  $\tilde{y}_T = (y_1, \dots, y_T)$ , given by  $L(\theta, \phi, \sigma^2 | \tilde{y}_T)$ . (exact likelihood)

# KALMAN FILTER



# Kalman Filter

*Kalman filter:*

- *purpose: to make inference about unobservable given the observable,*
- *application: signal extraction in engineering,*
- *economics: don't know the parameters  $F$ ,  $Q$ ,  $H$  and want to estimate them.*

## **State-space form**

ME: Measurement (Observation) equation:

$$y_t = H\beta_t + e_t, \quad e_t \sim N(0, R)$$

SE: Transition (State) equation:

$$\beta_t = \tilde{\mu} + F\beta_{t-1} + v_t, \quad v_t \sim N(0, Q),$$

$$E[e_t v_\tau'] = 0.$$

# Mean of $\beta$

- ①  $\beta_t$  is a random variable
  - it might be unobservable and no data for it,
  - it is normal random variable as it is sum of normal variables,  $v_t \sim N$ .

## Conditional mean

$$\beta_t | \Omega_{t-1} \sim N(E[\beta_t | \Omega_{t-1}], \text{var}(\beta_t | \Omega_{t-1}))$$

$$E[\beta_t | \Omega_{t-1}] = \beta_{t|t-1}, \quad \text{conditional expectations.}$$

- *We may not know what  $\beta$ 's are.*
- *If we have information about its distribution, we can calculate mean, variance, etc.*
- *$\beta_{t-1}$  may be not observable: take expectations of it*

$$E[\beta_t | \Omega_{t-1}] \equiv \beta_{t|t-1} = \tilde{\mu} + FE[\beta_{t-1} | \Omega_{t-1}] + 0$$

$$\beta_{t|t-1} = \tilde{\mu} + F\beta_{t-1|t-1},$$

- *In AR(1):  $E[y_t] = \mu + \phi E[y_{t-1}]$ , last term is observable.*

# Variance of $\beta$

## Conditional variance

$$\text{Var}(\beta_t | \Omega_{t-1}) \equiv P_{t|t-1} = E[(\beta_t - \beta_{t|t-1})(\beta_t - \beta_{t|t-1})'].$$

## Recall

$$\text{var}(ax) = a^2 \text{var}(x), \quad a - \text{scalar}, x - \text{random vector}.$$

Two sources of randomness (variation) for  $\beta_t$ :

- ①  $v_t$  is a random variable,
- ②  $\beta_{t-1}$  is also random so there might be difference between  $\beta_{t-1}$  and  $\beta_{t|t-1}$ , *there may not be equal to each other.*

$$P_{t|t-1} = F P_{t-1|t-1} F' + Q,$$

where  $P_{t|t-1}$ , uncertainty about  $\beta_t$  equals sum of uncertainty about  $\beta_{t-1}$ ,  $P_{t-1|t-1}$ , and uncertainty about  $v_t$ .

Note:  $\text{cov}(\beta_{t-1}, v_t) = 0$ .

$y_t$ 

- ②  $y_t$  is a random variable.
  - Now, we have data on  $y_t$ .
  - We have some joint density of  $y_t, \beta_t$  and some prior.
  - Using data we get posterior of  $\beta_t$ .
- *We want to make inference for  $\beta_t$  which we don't observe.*
- *We see  $y_t$  which is related to  $\beta_t$ .*
- *We make inferences on  $\beta_t$  by observing joint density (distribution) of  $y_t$  and  $\beta_t$  (Bayesian view).*

# Distribution of $y_t$

Distribution of  $y_t$  given state-space

$$y_t | \Omega_{t-1} \sim N(E[y_t | \Omega_{t-1}], \text{var}(y_t | \Omega_{t-1})),$$

- Conditional mean

$$E[y_t | \Omega_{t-1}] \equiv y_{t|t-1} = H\beta_{t|t-1} + 0$$

- Conditional variance

$$\text{var}(y_t | \Omega_{t-1}) \equiv f_{t|t-1} = HP_{t|t-1}H' + R,$$

*since we don't know  $\beta_t$ .*

- **Note:**  $\text{cov}(H\beta_t, e_t) = 0$  because  $E[v_t e_t] = 0$ .

*If  $E[v_t e_t] \neq 0$  we will add another term in the  $\text{var}(y_t | \Omega_{t-1})$  capturing that.*

# Joint Distribution

- Covariance between  $\beta_t$  and  $y_t$ :

$$\text{cov}(y_t, \beta_t | \Omega_{t-1}) = P_{t|t-1} H',$$

$$\text{as } \text{cov}(H\beta_t + e_t, \beta_t) = \text{cov}(H\beta_t, \beta_t) + \text{cov}(e_t, \beta_t) = \text{cov}(\beta_t, \beta_t) H' + 0.$$

Then, the joint distribution for  $y_t$  and  $\beta_t$  is joint normal:

$$\begin{pmatrix} \beta_t \\ y_t \end{pmatrix} \bigg| \Omega_{t-1} \sim N \left( \begin{bmatrix} \beta_{t|t-1} \\ H\beta_{t|t-1} \end{bmatrix}, \begin{bmatrix} P_{t|t-1} & P_{t|t-1} H' \\ P_{t|t-1} H' & f_{t|t-1} \end{bmatrix} \right).$$

# Kalman Filter

Two steps of Kalman Filter :

- (a) Prediction,
- (b) Given  $y_t$  updating inference on  $\beta_t$ .

## Definition

Given  $\beta_{0|0}, P_{0|0}$ , Kalman Filter solves the following six equations for  $i = 1, \dots, T$

Prediction of  $y_t, \beta_t$

$$(1) \quad \beta_{t|t-1} = \tilde{\mu} + F\beta_{t-1|t-1},$$

$$(2) \quad P_{t|t-1} = F P_{t-1|t-1} F' + Q,$$

Forecast error:

$$(3) \quad \eta_{t|t-1} \equiv y_t - y_{t|t-1} = y_t - H\beta_{t|t-1},$$

Variance of forecast error:

$$(4) \quad f_{t|t-1} = H P_{t|t-1} H' + R$$

Updating of  $y_t, \beta_t$

$$(5) \quad \beta_{t|t} = \beta_{t|t-1} + \kappa_t \eta_{t|t-1},$$

$$(6) \quad P_{t|t} = P_{t|t-1} - \kappa_t H P_{t|t-1},$$

$$\kappa_t \equiv P_{t|t-1} H' f_{t|t-1}^{-1} \quad \text{“Kalman gain”}.$$

# Kalman Filter

- $\beta_{0|0}, P_{0|0}$ , are equal to unconditional mean and variance, and reflect prior beliefs.
- If the state space model is covariance stationary,

$$\begin{aligned}
 E[\beta] = \beta_{0|0} &= (\mathbb{I} - \mu)^{-1} \tilde{\mu} \\
 \text{var}(\beta) = P_{0|0} &= FP_{0|0}F' + Q \\
 \text{vec}(P_{0|0}) &= \text{vec}(FP_{0|0}F') + \text{vec}(Q) \\
 \text{vec}(P_{0|0}) &= (F \otimes F)\text{vec}(P_{0|0}) + \text{vec}(Q) \\
 \text{vec}(P_{0|0}) &= (\mathbb{I} - F \otimes F)^{-1}\text{vec}(Q).
 \end{aligned}$$

since  $\text{vec}(ABC) = (C' \otimes A)\text{vec}(B)$ .

- Equation (5) is a linear combination of previous guess and forecast error.

$$(5) \quad \beta_{t|t} = \beta_{t|t-1} + \kappa_t \eta_{t|t-1},$$

$$(6) \quad P_{t|t} = P_{t|t-1} - \kappa_t H P_{t|t-1},$$

$$\kappa_t \equiv P_{t|t-1} H' f_{t|t-1}^{-1} \quad \text{“Kalman gain”}.$$



# Kalman Gain

- The stronger the covariance between  $y_t$  and  $\beta_t$ , the more we will update when we see high forecast error.
- If the relationship is weaker, we don't put much weight as probably it is not driven by  $\beta_t$ .
- The weight depends on the variance of forecast error: if  $f^{-1}$  big, put high weight on that observations.
- Once we have  $\eta_{t|t-1}, f_{t|t-1}$ , we can do MLE after constructing the joint likelihood of prediction error decomposition.
  - *The Kalman gain depends on the relationship between  $y_t$  and  $\beta_t$  since  $P_{t|t-1}H' = cov(\beta_t, y_t)$  and  $f_{t|t-1}^{-1}$  is the precision of the forecast error.*
  - *The bigger the variance of forecast error the smaller the Kalman gain and less weight put to updating.*
  - *Equation (6) measures conditional variance.*
  - *Since we observe  $y_t$  the uncertainty declines.*