Wykład: Teoria wzrostu - model Solowa

Makroekonomia

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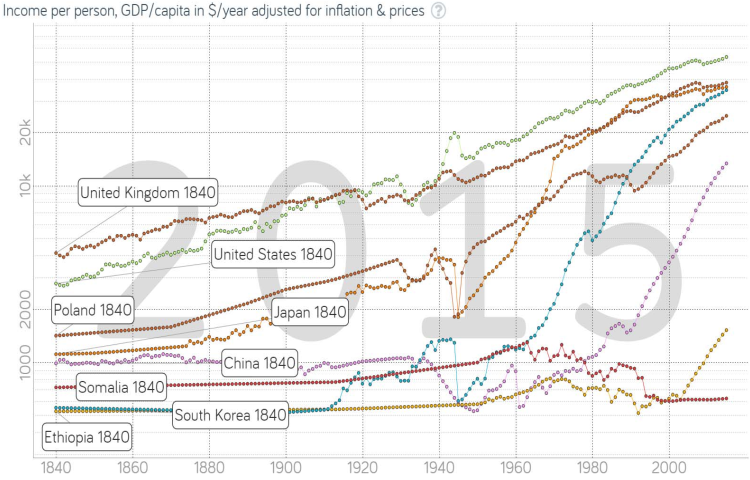
Once one starts to think about economic growth, it is hard to think about anything else.

Robert Lucas (1988)

Historical overview

- Starting from the nineteenth century, growth rates have been on average much higher than in the past
- Still, huge differences across countries, both in per capita income levels and growth rates:
  - *Growth miracles*: e.g. the four Asian tigers (Hong Kong, Singapore, South Korea, Taiwan)
  - *Growth disasters*: e.g. Argentina in the early twentieth century
- Subsaharian Africa ⇔ China

GDP growth *per capita*



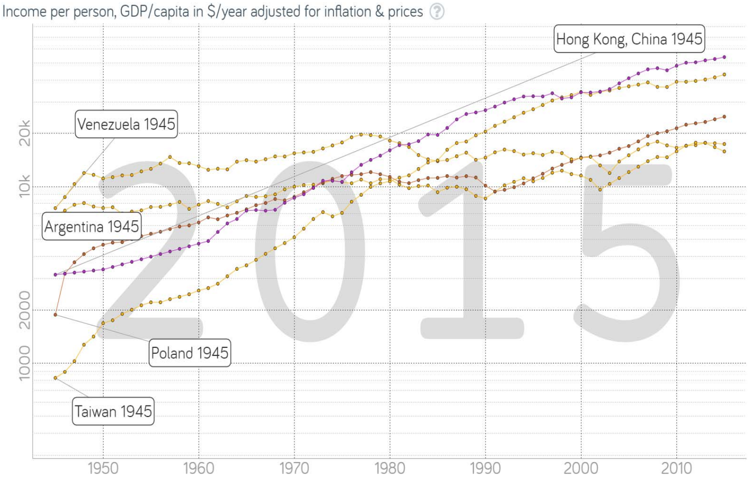
Miracles and disasters

- *South Korea*: Sustained growth of 6% since 1960
  - GDP doubles every 12 years
  - In 50 years: doubles 4 times
  - 2x2x2x2 = 16 times richer than grandparents!!

- Contrast with *Venezuela*
  - Overall negative growth for a period of 40 years
  - Grandchildren poorer than grandparents

- Small differences in growth rates compound over time to generate enormous differences in incomes
- Not everybody stays on the growth path

GDP growth *per capita*



Kaldor's stylized facts

- Fact 1: Output per capita (Y/L) and capital intensity (K/L) keep increasing
- Fact 2: The capital output ratio (K/Y) is roughly constant
- Fact 3: Hourly wages keep rising
- Fact 4: The rate of return to capital is constant
- Fact 5: The relative shares of GDP going to labor and capital are constant

Solow-Swan model

- The Solow/Solow-Swan model is the first systematic attempt to formally model the process of economic growth
- Named after Robert Solow (MIT, Nobel Prize winner 1987 for his contribution to economic growth) and Trevor Swan (ANU)
- The Solow model is similar to modern macro models. It is a dynamic equilibrium model, but it assumes arbitrary decision rules for the household.

Solow-Swan model

- Benchmark model of economic growth, capital accumulation
- Simple setting: no government, closed economy, full employment, homogenous goods, identical firms, identical households
- Key element: aggregate production function with smooth substitution between capital and labor
- Key conclusion: sustained growth only through productivity growth, not capital accumulation
- In the class: discrete time, $t = 0, 1, 2, \ldots$
- In the Solow model the social planner allocation can be decentralized. Today we consider the decentralized market allocation.

Representative household

- In the Solow model household does not solve maximization problem
- Household owns both capital and labor and rents them to firms at the real wage rate w_t and the rental rate r_t
  - Labor is supplied inelastically and grows at a rate n

$$L_{t+1} = (1+n)L_t, \qquad 1+n > 0, \quad L_0 > 0$$

  - The capital stock evolves with investment (I_t) and is subject to depreciation (rate δ)

$$K_{t+1} = (1-\delta)K_t + I_t, \qquad 0 < \delta < 1, \quad K_0 > 0$$

- Household uses income to finance either consumption or investment in new capital
- Household splits their income into consumption (C) and savings (S)

$$w_t L_t^s + r_t K_t^s = C_t + S_t = C_t + I_t$$

and chooses to save a constant fraction s of their income,

  - no optimization

Firm(s)

- Firms are competitive
  - we assume they are identical and consider a single representative firm

- Output Y_t is produced by the firm with physical capital, labor, and productivity A_t according to the aggregate production function

$$Y_t = F(K_t, A_t L_t)$$

i.e. productivity is labor-augmenting (or Harrod-neutral)

- Productivity A_t grows exogenously

$$A_{t+1} = (1+g)A_t, \qquad 1+g > 0, \quad A_0 > 0$$

and is available for free.

Production function

- A two-input (x_1, x_2) production function $F(x_1, x_2)$ is assumed to
  - be increasing in each input, $\frac{\partial F(x_1,x_2)}{\partial x_i} = F_{x_i}(x_1, x_2) > 0$, $i = 1, 2$.
  - have diminishing returns to each input, $F_{x_i x_i}(x_1, x_2) < 0$
  - have constant returns to scale, $F(cx_1, cx_2) = cF(x_1, x_2)$

- We will assume that
  - $F(x_1, 0) = F(0, x_2) = 0$, i.e. all factors are essential
  - $F(x_1, x_2)$ satisfies Inada conditions, i.e. $\forall x_1, x_2 > 0$

$$F_{x_1}(0, x_2) = \infty, \quad F_{x_1}(\infty, x_2) = 0,$$
$$F_{x_2}(x_1, 0) = \infty, \quad F_{x_2}(x_1, \infty) = 0$$

Firm's maximization problem

- Every period firm chooses inputs to maximize profits

$$\max_{K_t^d, L_t^d} \left\{ Y_t - w_t L_t^d - r_t K_t^d \right\}$$

taking w_t and r_t as given.

- Using $Y_t = F(K_t, A_t L_t)$ the first order conditions are

$$\frac{\partial F(K_t, A_t L_t)}{\partial K_t} = r_t$$
$$\frac{\partial F(K_t, A_t L_t)}{\partial L_t} = w_t$$

Market clearing

- The firm's demand for labor equals the supply of labor from households, $L_t^s = L_t^d$
- Capital supplied by household equals capital input demanded by firm, $K_t^s = K_t^d$
- Every period output is either consumed or invested

$$Y_t = C_t + I_t$$

income is either consumed or saved $\equiv$ investment equals saving

$$C_t + I_t = C_t + S_t \Rightarrow I_t = S_t$$

- Key assumption: household save a constant fraction of their income, $S_t = sY_t, \Rightarrow I_t = sY_t$

Competitive equilibrium

Definition

A competitive equilibrium of the economy is a sequence of allocations $\{K_t^s, C_t, I_t, K_t^d, L_t^d\}_{t=0}^{\infty}$ *and prices* $\{w_t, r_t\}_{t=0}^{\infty}$ *such that given initial* $K_0 > 0, L_0 > 0, A_0 > 0$ *and the dynamics of* A_t *and* L_t^s *and*

(i) *given the* $\{w_t, r_t\}_{t=0}^{\infty}$ *the sequence of* $\{K_t^s, C_t, I_t\}_{t=0}^{\infty}$ *is consistent with the behavior of households,*

(ii) $\{K_t^d, L_t^d\}$ *maximizes firms profits for every t,*

(iii) *the goods, capital and labor markets clear every period.*

- The equilibrium exists, is unique (in terms of aggregate quantities), and corresponds to social planner problem/centralized allocation.

Intensive form

- We can express all variables of the model in terms of efficiency units

$$y_t \equiv \frac{Y_t}{A_t L_t}, \quad k_t \equiv \frac{K_t}{A_t L_t}, \quad c_t = \frac{C_t}{A_t L_t}, \dots$$

- Since $F(\cdot,\cdot)$ is assumed to have constant returns to scale then

$$y_t = \frac{Y_t}{A_t L_t} = \frac{F(K_t, A_t L_t)}{A_t L_t} = F\left(\frac{K_t}{A_t L_t}, \frac{A_t L_t}{A_t L_t}\right) = F\left(\frac{K_t}{A_t L_t}, 1\right) = F(k_t, 1) \equiv f(k_t)$$

- $f(k_t)$ is the intensive form of the production function and has the following properties
  - the marginal product of k is positive, $f'(k_t) > 0$
  - ... but MPk is diminishing, $f''(k_t) < 0$
  - it satisfies the Inada condition: $\lim_{k\to 0} f'(k) = \infty$, $\lim_{k\to\infty} f'(k) = 0$

Example: Cobb-Douglas

- Example: Cobb-Douglas production function

$$F(K_t, A_t L_t) = K_t^{\alpha}(A_t L_t)^{1-\alpha}$$

where α is the elasticity of output with respect to capital.

  - F is increasing, features diminishing marginal product of capital, has CRS, all factors are essential, satisfies Inada conditions.

- The intensive form $f(k_t)$

$$y_t = \frac{K_t^{\alpha}(A_t L_t)^{1-\alpha}}{A_t L_t} = \left(\frac{K_t}{A_t L_t}\right)^{\alpha} = k_t^{\alpha} = f(k_t)$$

for $k = \frac{K}{AL}$

  - f is increasing, features diminishing marginal product of capital, has DRS, satisfies Inada conditions.

Evolution of k_t

- Dynamics of capital per efficiency unit can be written as

$$\begin{aligned} k_{t+1} = \frac{K_{t+1}}{A_{t+1}L_{t+1}} &= \frac{(1-\delta)K_t + sF(K_t, A_tL_t)}{A_{t+1}L_{t+1}} \\ &= (1-\delta)\frac{K_t}{A_{t+1}L_{t+1}} + s\frac{F(K_t, A_tL_t)}{A_{t+1}L_{t+1}} \\ &= (1-\delta)\frac{K_t}{(1+g)A_t(1+n)L_t} + s\frac{F(K_t, A_tL_t)}{(1+g)A_t(1+n)L_t} \\ &= \frac{1-\delta}{(1+g)(1+n)}k_t + \frac{s}{(1+g)(1+n)}f(k_t) = \psi(k_t) \end{aligned}$$

- this is an autonomous difference equation in k_t

Steady state

- The steady state level of capital per efficiency units k^* is given by the condition $k_{t+1} = k_t = k^*$.
- Given the difference equation on the previous slide k^* solves $k^* = \psi(k^*)$, i.e.

$$k^* = \frac{1-\delta}{(1+g)(1+n)}k^* + \frac{s}{(1+g)(1+n)}f(k^*)$$

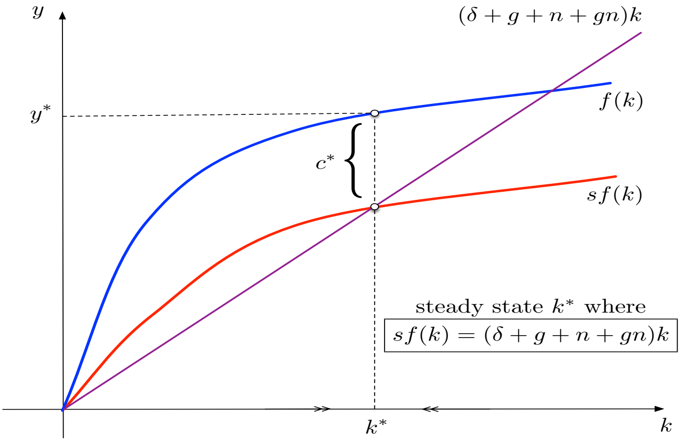
or

$$sf(k^*) = (g + n + \delta + g \cdot n)k^*$$

- k^* depends on parameters of the model $s, g, n, \delta, f(\cdot)$.
- Other variables in the steady state can be computed accordingly

$$y^* = f(k^*), \quad c^* = (1-s)y^*, \quad i^* = sy^*, \ldots$$

Steady state



Dynamics of k_t

- We can analyze the behavior of k_t by looking at $\Delta k_{t+1} = k_{t+1} - k_t$
  1. In the steady state $\Delta k_t = 0$ for all t
  2. k_t increases if $k_t < k^*$ as

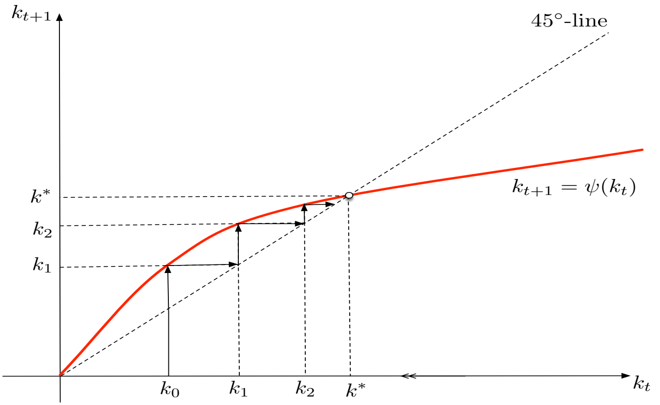
$$
\begin{aligned}
k_{t+1} > k_t \iff & \frac{1-\delta}{(1+g)(1+n)} k_t + \frac{s}{(1+g)(1+n)} f(k_t) > k_t \\
\iff & (1-\delta)k_t + sf(k_t) > (1+g)(1+n)k_t \\
\iff & sf(k_t) > (g+n+\delta+g\cdot n)k_t \\
\iff & k_t < \frac{sf(k_t)}{(g+n+\delta+g\cdot n)} < \frac{sf(k^*)}{(g+n+\delta+g\cdot n)} = k^*
\end{aligned}
$$

  3. k_t decreases if $k_t > k^*$

$$
k_{t+1} < k_t \iff sf(k_t) < (g+n+\delta+g\cdot n)k_t \iff k_t > k^*
$$

- (2) and (3) together imply that for any $k_0 > 0$, k_t converges to k^*, i.e. $k_t \to k^*$.

Phase diagram



Stability of k^*

- We can determine the local stability of a steady state.
- We want to see how $k_{t+1} = \psi(k_t)$ behaves around k^*.
- Since $k_{t+1} = \psi(k_t)$ is nonlinear difference equation we look at its local stability around the steady state, k^*.
- If we linearize $\psi(k_t)$ around $k_t = k^*$ (first-order Taylor expansion) we get

$$k_{t+1} = \psi(k_t) \approx \psi(k_t)\Big|_{k_t=k^*} + \psi'(k_t)\Big|_{k_t=k^*}(k_t - k^*)$$
$$k_{t+1} = \psi(k^*) + \psi'(k^*)(k_t - k^*)$$

which, using the steady state condition $\psi(k^*) = k^*$, can be written as

$$k_{t+1} - k^* = \psi'(k^*)(k_t - k^*)$$

- The above equation is a linear difference equation and it is stable if $|\psi'(k^*)| < 1$.

Stability of k^*

- We can compute $\psi'(k^*)$ as follows

$$\psi(k_t) = \frac{s}{(1+g)(1+n)} f(k_t) + \frac{1-\delta}{(1+g)(1+n)} k_t$$
$$\psi'(k^*) = \frac{sf'(k^*)}{(1+g)(1+n)} + \frac{1-\delta}{(1+g)(1+n)}$$

- The stability is govern by the condition $\psi'(k^*) < 1$, i.e.,

$$\psi'(k^*) < 1 \iff \frac{sf'(k^*)}{(1+g)(1+n)} + \frac{1-\delta}{(1+g)(1+n)} < 1$$
$$\Rightarrow sf'(k^*) < g + n + \delta + gn$$

- Because of the Inada conditions the steady state with $k^* > 0$ is stable.

Growth rate in the steady state

- Recall that in the steady state $\psi(k_t) = k_{t+1} = k_t = k^*$, i.e., k^* is the solution to the fixed point problem

$$k^* = \psi(k^*)$$

- The growth rate of k_t at the steady state can be computed as

$$g_k = \frac{\Delta k_{t+1}}{k_t}\bigg|_{k_t=k^*} = \frac{0}{k^*} = 0$$

- The growth rate of output per unit of effective labor, $y = \frac{Y}{AL}$, is

$$g_y = \frac{\Delta y_{t+1}}{y_t}\bigg|_{k_t=k^*} = \frac{f(k_{t+1}) - f(k_t)}{f(k_t)}\bigg|_{k_t=k^*} = \frac{f(k^*) - f(k^*)}{f(k^*)} = 0$$

- The consumption and investment per unit of effective labor can be all computed in a similar way as $c = (1-s)y$, $i = sy \Rightarrow g_c = g_i = 0$

Growth rate of aggregate capital stock

- The growth rate of aggregate capital stock, K_t, can be computed using $K_t = k_t A_t L_t$

$$g_K = \left.\frac{\Delta K_{t+1}}{K_t}\right|_{k_t=k^*} = \left.\frac{K_{t+1}}{K_t}\right|_{k_t=k^*} - 1 = \left.\frac{k_{t+1}A_{t+1}L_{t+1}}{k_t A_t L_t}\right|_{k_t=k^*} - 1$$
$$= \frac{k^* A_{t+1} L_{t+1}}{k^* A_t L_t} - 1 = \frac{A_{t+1}}{A_t}\frac{L_{t+1}}{L_t} - 1$$
$$= (1+g)(1+n) - 1 = n + g + ng \approx n + g$$

  - for small value of n and g the product $n \cdot g$ is very small

Growth rate of aggregate variables

- Similarly,

$$g_Y = \left.\frac{\Delta Y_{t+1}}{Y_t}\right|_{k_t=k^*} = \frac{y^* A_{t+1} L_{t+1}}{y^* A_t L_t} - 1 = (1+g)(1+n) - 1 \approx n + g$$

$$g_C = \left.\frac{\Delta C_{t+1}}{C_t}\right|_{k_t=k^*} = \frac{(1-s) y^* A_{t+1} L_{t+1}}{(1-s) y^* A_t L_t} - 1 = (1+g)(1+n) - 1 \approx n + g$$

$$g_I = \left.\frac{\Delta I_{t+1}}{I_t}\right|_{k_t=k^*} = \frac{s y^* A_{t+1} L_{t+1}}{s y^* A_t L_t} - 1 = (1+g)(1+n) - 1 \approx n + g$$

Some other rates of growth at the steady state

- We can also compute the growth rate of GDP per capita, $\frac{Y}{L}$

$$g_{Y/L} = \left.\frac{\Delta(Y_{t+1}/L_{t+1})}{(Y_t/L_t)}\right|_{k_t=k^*} = \left.\frac{(Y_{t+1}/L_{t+1})}{(Y_t/L_t)}\right|_{k_t=k^*} - 1 = \left.\frac{\frac{Y_{t+1}}{Y_t}}{\frac{L_{t+1}}{L_t}}\right|_{k_t=k^*} - 1$$
$$= \frac{(1+g)(1+n)}{(1+n)} - 1 = g$$

- Similarly, $\frac{C}{L}$, $\frac{I}{L}$ and $\frac{K}{L}$ all grow at the rate of growth of technology g.

- Note that in the steady state some variables are not constant but grow at constant rates. We say that they are on the *balanced growth path*.

Wages and rental rate

- How about the rental rate r and the wage w?
- Take the first order conditions of the firm's problem:

$$w_t = \frac{\partial F(K_t, A_t L_t)}{\partial L_t} \quad \text{and} \quad r_t = \frac{\partial F(K_t, A_t L_t)}{\partial K_t} = F_K(K_t, A_t L_t),$$

as well as $F_K(K, AL) = f'(k)$ and

$$\begin{aligned}\frac{\partial F(K, AL)}{\partial L} &= \frac{\partial \left[F(\frac{K}{AL}, 1)AL\right]}{\partial L} \\ &= -F_K\left(\frac{K}{AL}, 1\right)\frac{K}{A}\frac{1}{L^2} \cdot AL + F\left(\frac{K}{AL}, 1\right) \cdot A = A \cdot \left[f(k) - f'(k)k\right]\end{aligned}$$

Wages and rental rate

- How about rental rate r and wage w?

- The growth rate of r and w at the balanced growth path equals

$$g_w = \left.\frac{w_{t+1}}{w_t}\right|_{k_t=k^*} - 1 = \frac{A_{t+1}\left[f(k^*) - f'(k^*)k^*\right]}{A_t\left[f(k^*) - f'(k^*)k^*\right]} - 1 = 1 + g - 1 = g$$

$$g_r = \left.\frac{r_{t+1}}{r_t}\right|_{k_t=k^*} - 1 = \frac{f'(k^*)}{f'(k^*)} - 1 = 0$$

$\Longrightarrow$ Wage grows at a constant rate but r_t is constant.

Probing the steady state

- Recall that the steady state values of k^* (and y^*, c^*, i^*, r^*) are determined by the parameters of the model s, g, n, δ and the shape of the production function f:

$$\frac{sf(k^*)}{k^*} = g + n + \delta + gn$$

  - g, δ, and $f(\cdot)$ are related to technology
  - saving rate s is decided (exogenously) by household
- To compute the effect on k^* of a permanent change in s, total differentiate the equation for k^*

$$ds\frac{f(k^*)}{k^*} + \frac{s}{k^*}f'(k^*)dk - \frac{sf(k^*)}{(k^*)^2}dk = 0$$

which yields

$$\frac{\partial k^*}{\partial s} = \frac{dk^*}{ds} = \frac{k^*}{s}\frac{1}{1-\alpha(k^*)} > 0,$$

where $\alpha(k) = \frac{kf'(k)}{f(k)}$ is a capital share in output at k^*

  - $\alpha(k) = \frac{r \cdot K}{Y} = \frac{F_K \cdot K}{Y} = \frac{f'(k)k}{y}$

Increasing saving rate

- The effect of s on y^* is

$$\frac{\partial y^*}{\partial s} = \frac{\partial f(k^*)}{\partial s} = f'(k^*)\frac{\partial k^*}{\partial s} = f'(k^*)\frac{k^*}{s}\frac{1}{1-\alpha(k^*)} > 0$$

- Computing the elasticity of output with respect to the saving rate,

$$\frac{\partial y^*}{\partial s}\frac{s}{y^*} = \frac{\alpha(k^*)}{1-\alpha(k^*)}$$

we see that the steady-state effects of saving on output depends positively on capital share in output.

Golden rule of capital accumulation

- We (Solow) assumed that s is chosen exogenously but we saw that higher s implies higher k^*, y^*.
- How about c^*?

$$c^* = (1-s)y^*$$

  - ambiguous effect of s on c^*
- What is the savings rate maximizing the steady-state consumption?

$$\max_s \{(1-s)y^*\}$$

- We can solve it directly (take FOC) as

$$(1-s)f'(k^*)\frac{\partial k^*}{\partial s} - f(k^*) = 0 \Rightarrow s^{GR} = \alpha(k^*)$$

- The consumption is the highest for $s = \alpha(k^*)$.

Golden rule of capital accumulation

- Note that

$$c^* = (1-s)f(k^*) = f(k^*) - sf(k^*) = f(k^*) - (g+n+\delta+gn)k^*$$

- We could calculate the maximum consumption by solving

$$\max_{k^*}\{f(k^*) - (g+n+\delta+gn)k^*\}$$

which yield familiar first order condition

$$f'(k^*) = g+n+\delta+gn$$

and

$$k^{GR} = (f')^{-1}(g+n+\delta+gn)$$

Dynamic inefficiency

- Since the savings rate is exogenous, there is no reason to expect that the capital per efficiency unit will settle at the golden-rule level (s^{GR}) in the Solow model.

- Dynamic inefficiency

  If $s > s^{GR}$ (equivalently, $k^* > k^{GR}$), the economy is dynamically inefficient: if the saving rate is lowered to $s = s^{GR}$ for all t, then consumption in all periods will be higher!

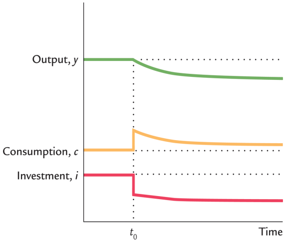
- Dynamic efficiency

  If $s < s^{GR}$ (equivalently, $k^* < k^{GR}$), then raising s towards s^{GR} will increase consumption in the long run, but at the expense of lower consumption in the short run;

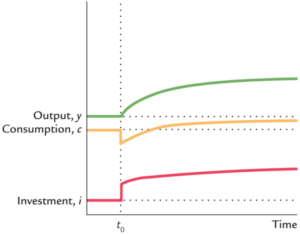
  - whether such a trade-off is desirable depends on how one weighs current generations vis-a-vis future generations

Transition to Golden Rule

- Paths of y, c and i during the transition to the steady state



Dynamic inefficiency



Dynamic efficiency

Conditional convergence

- The dynamics of the model are given by

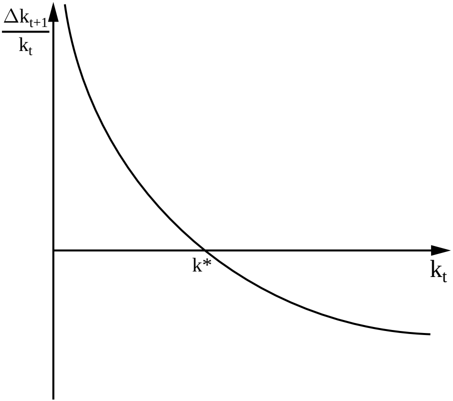
$$k_{t+1} = \psi(k_t) = \frac{1-\delta}{(1+g)(1+n)}k_t + \frac{s}{(1+g)(1+n)}f(k_t)$$

or, in terms of the rate of growth, by

$$\frac{\Delta k_{t+1}}{k_t} = \frac{1}{(1+g)(1+n)}\left[\frac{sf(k_t)}{k_t} - (g+n+\delta+ng)\right]$$

- At any point in time the growth rate of the capital/output depends on
  - parameters of the model
  - the distance from the steady state (k^*, y^*)
- The further away from the steady state the faster the rate of convergence

Convergence



Convergence in the data: US states

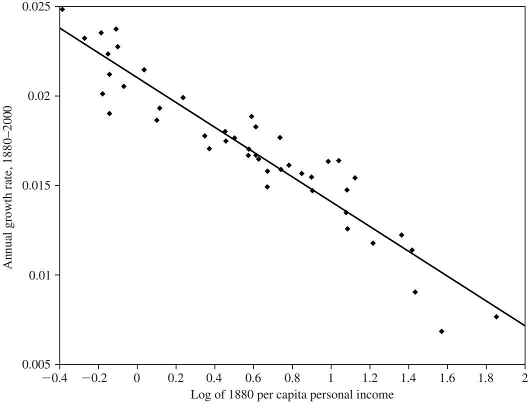


Figure 11.2
Convergence of personal income across U.S. states: 1880 personal income and 1880–2000 income growth.
The average growth rate of state per capita income for 1880–2000, shown on the vertical axis, is negatively related to the log of per capita income in 1880, shown on the horizontal axis. Thus, absolute β convergence exists for the U.S. states.

Barro, Sala-i-Martin (2004)

Convergence in the data: Japan

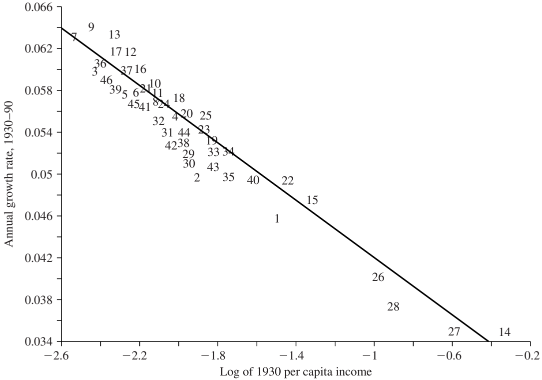


Figure 11.5
Convergence of personal income across Japanese prefectures: 1930 income and 1930–90 income growth. The growth rate of prefectural per capita income for 1930–90, shown on the vertical axis, is negatively related to the log of per capita income in 1930, shown on the horizontal axis. Thus absolute β convergence exists for the Japanese prefectures. The numbers shown identify each prefecture; see table 11.10.

Barro, Sala-i-Martin (2004)

Convergence in the data: European regions

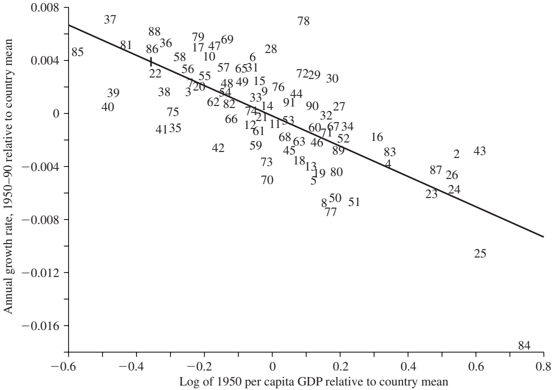


Figure 11.8
Growth rate from 1950 to 1990 versus 1950 per capita GDP for 90 regions in Europe. The growth rate of a region's per capita GDP for 1950–90, shown on the vertical axis, is negatively related to the log of per capita GDP in 1950, shown on the horizontal axis. The growth rate and level of per capita GDP are measured relative to the country means. Hence, this figure shows that absolute β convergence exists for the regions within Germany, the United Kingdom, Italy, France, the Netherlands, Belgium, Denmark, and Spain. The numbers shown identify the regions; see table 11.9.

Barro, Sala-i-Martin (2004)

No convergence in the data: country-level data

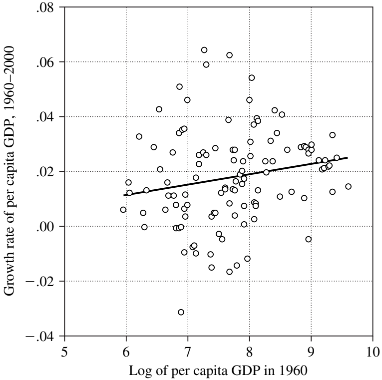


Figure 12.2
Growth rate versus GDP (simple relation). These data are for the 112 countries described in figure 12.1. The log of per capita GDP in 1960 is on the horizontal axis, and the growth rate of per capita GDP from 1960 to 2000 is on the vertical. The correlation between the two is weakly positive: 0.19. Thus there is no evidence from the broad cross-country sample of absolute convergence.

Barro, Sala-i-Martin (2004)

Conditional convergence: regression

Table 12.3
Basic Cross-Country Growth Regressions

(1) Explanatory Variable	(2) Coefficient	(3) Coefficient for Low-Income Sample	(4) Coefficient for High-Income Sample	(5) p Value[a]	(6) Coefficient with Data at 5-Year Intervals
Log of per capita GDP	−0.0248 (0.0029)	−0.0207 (0.0052)	−0.0318 (0.0049)	0.12	−0.0237 (0.0029)
Male upper-level schooling	0.0036 (0.0016)	0.0056 (0.0045)	0.0020 (0.0016)	0.44	0.0023 (0.0015)
1/(life expectancy at age 1)	−5.04 (0.86)	−5.13 (1.18)	−1.28 (1.44)	0.040	−4.91 (0.90)
Log of total fertility rate	−0.0118 (0.0050)	−0.0209 (0.0120)	−0.0211 (0.0054)	0.99	−0.0160 (0.0048)
Government consumption ratio	−0.062 (0.023)	−0.102 (0.031)	−0.000 (0.031)	0.021	−0.066 (0.021)
Rule of law	0.0185 (0.0059)	0.0237 (0.0099)	0.0223 (0.0063)	0.90	0.0174 (0.0062)
Democracy	0.079 (0.028)	0.044 (0.049)	0.105 (0.038)	0.32[b]	0.032 (0.017)
Democracy squared	−0.074 (0.025)	−0.054 (0.052)	−0.080 (0.031)	0.67	−0.028 (0.016)
Openness ratio	0.0054 (0.0048)	0.0169 (0.0113)	0.0061 (0.0046)	0.38	0.0094 (0.0043)
Change in terms of trade	0.130 (0.053)	0.181 (0.076)	0.036 (0.070)	0.16	0.029 (0.021)
Investment ratio	0.083 (0.024)	0.109 (0.035)	0.077 (0.027)	0.46	0.058 (0.022)
Inflation rate	−0.019 (0.010)	−0.019 (0.012)	−0.019 (0.009)	0.99	−0.031 (0.007)
Constant	0.296 (0.034)	0.294 (0.052)	0.295 (0.052)	0.99[c]	0.306 (0.035)
Dummy, 1975–85	−0.0078 (0.0026)	−0.0078 (0.0038)	−0.0066 (0.0032)	0.81	[d]
Dummy, 1985–95	−0.0128 (0.0034)	−0.0194 (0.0051)	−0.0052 (0.0040)	0.031	
Number of observations	72, 86, 83	26, 38, 33	46, 48, 50		72, 79, 86, 84 79, 80, 60
R-squared	.60, .49, .51	.78, .53, .65	.56, .56, .40		.40, .26, .27, .31, .46, .19, .04

Notes: Estimation is by three-stage least squares. In column 2 the dependent variables are the growth rates of per capita GDP for 1965–75, 1975–85, and 1985–95. Instruments are the values in 1960, 1970, and 1980 of the log of per capita GDP, the life-expectancy variable, and the fertility variable; averages for 1960–64, 1970–74, and 1980–84 of the government consumption variable and the investment ratio; values in 1965, 1975, and 1985 of the schooling variable and the democracy variables; the openness and terms-of-trade variables (growth rates over 1965–75, 1975–85, and 1985–95, interacted with the corresponding averages of the ratio of exports plus imports to GDP); and dummies for Spanish or Portuguese colonies and other colonies (aside from Britain and France). The variances of the error terms are allowed to be correlated over the time periods and to have different variances for each period. Columns 3 and 4 separate the samples into countries with levels of per capita GDP below and above the median (for 1960, 1970, and 1980). Column 6 uses equations for economic growth for seven five-year periods, 1965–70, . . . , 1995–2000.

Barro, Sala-i-Martin (2004)

Conditional convergence: 112 countries

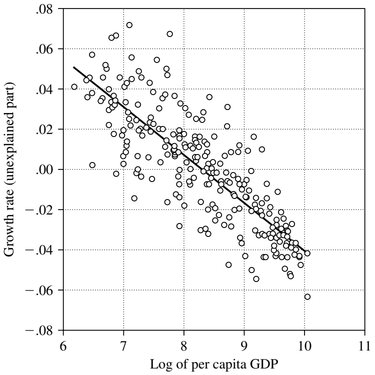


Figure 12.3
Growth rate versus GDP (partial relation). The log of per capita GDP for 1965, 1975, and 1985 is shown on the horizontal axis. The vertical axis plots the corresponding growth rate of real per capita GDP from 1965 to 1975, 1975 to 1985, and 1985 to 1995. These growth rates are filtered for the estimated effect of the explanatory variables other than the log of per capita GDP that are shown in column 2 of table 12.3. The filtered values were then normalized to have zero mean. Thus the diagram shows the partial relation between the growth rate of per capita GDP and the log of per capita GDP.

Barro, Sala-i-Martin (2004)

Conditional convergence: education

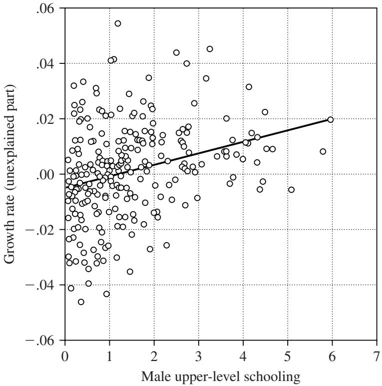


Figure 12.4
Growth rate versus schooling (partial relation). The diagram shows the partial relation between the growth rate of per capita GDP and the average years of school attainment of males at the upper level (higher schooling plus secondary schooling). The variable on the horizontal axis is measured in 1965, 1975, and 1985. See the description of figure 12.3 for the general procedure.

Barro, Sala-i-Martin (2004)

Conditional convergence: life expectancy

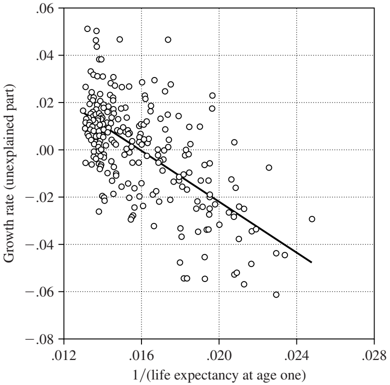


Figure 12.5
Growth rate versus life expectancy (partial relation). The diagram shows the partial relation between the growth rate of per capita GDP and the reciprocal of life expectancy at age one. The variable on the horizontal axis is measured in 1960, 1970, and 1980. See the description of figure 12.3 for the general procedure.

Barro, Sala-i-Martin (2004)

Conditional convergence: fertility

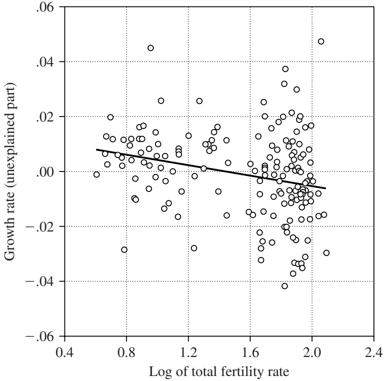


Figure 12.6
Growth rate versus fertility rate (partial relation). The diagram shows the partial relation between the growth rate of per capita GDP and the log of the total fertility rate. The variable on the horizontal axis is measured in 1960, 1970, and 1980. See the description of figure 12.3 for the general procedure.

Barro, Sala-i-Martin (2004)

Conditional convergence: institutions - rule of law

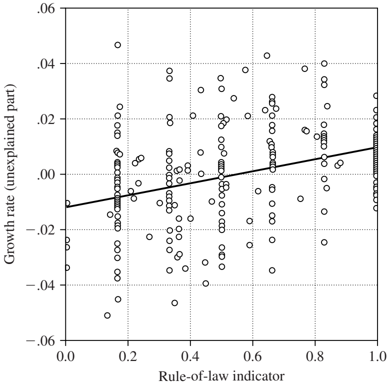


Figure 12.8
Growth rate versus rule of law (partial relation). The diagram shows the partial relation between the growth rate of per capita GDP and the Political Risk Services indicator for maintenance of the rule of law. The variable on the horizontal axis associated with growth in 1965–75 and 1975–85 applies to 1982 or 1985. The value associated with growth in 1985–95 is the average for 1985–94. See the description of figure 12.3 for the general procedure.

Barro, Sala-i-Martin (2004)

Conditional convergence: investment

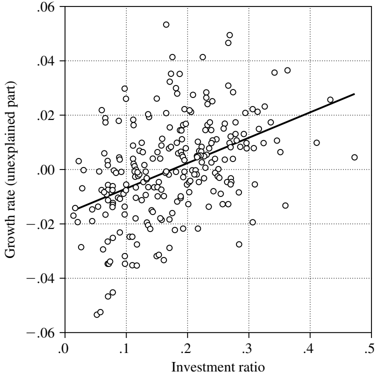


Figure 12.12
Growth rate versus investment (partial relation). The diagram shows the partial relation between the growth rate of per capita GDP and the ratio of investment to GDP. The variable on the horizontal axis is the average for 1965–74, 1975–84, and 1985–94. See the description of figure 12.3 for the general procedure.

Barro, Sala-i-Martin (2004)

Conditional convergence: openness

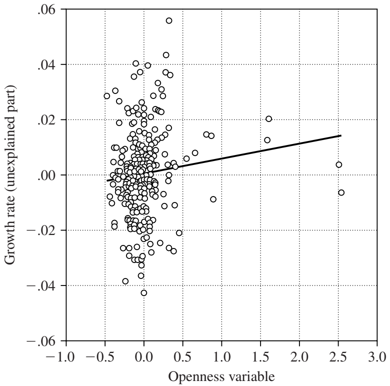


Figure 12.10
Growth rate versus openness (partial relation). The diagram shows the partial relation between the growth rate of per capita GDP and the openness ratio. This variable is the ratio of exports plus imports to GDP, filtered for the usual relation of this ratio to the logs of population and area. The variable on the horizontal axis is the average for 1965–74, 1975–84, and 1985–94. See the description of figure 12.3 for the general procedure.

Barro, Sala-i-Martin (2004)

Speed of convergence

- We know that for any $k_0 > 0$ the capital per efficiency unit reaches eventually the steady state but how quickly?
- Recall that the linearized version can be written as

$$k_{t+1} - k^* = \psi'(k^*)(k_t - k^*)$$

Since

$$k_t - k^* = \psi'(k^*)(k_{t-1} - k^*)$$

iterating this equation backward yields

$$k_t - k^* = [\psi'(k^*)]^t (k_0 - k^*)$$

$\Rightarrow$ the higher the $\psi'(k^*)$ the slower capital converges to k^*.

- We can compute the half-life of convergence, that is how long it takes to close half the gap between k_0 and k^*:

$$k_\tau - k^* = \frac{1}{2}(k_0 - k^*) \Rightarrow [\psi'(k^*)]^\tau = \frac{1}{2}$$

Taking logs of last equation and solving it yield

$$\tau = \frac{\log(1/2)}{\log(\psi'(k^*))}$$

Speed of convergence

- As
$$\psi'(k^*) = \frac{sf'(k^*)}{(1+g)(1+n)} + \frac{1-\delta}{(1+g)(1+n)}$$
the speed of convergence depends on the parameters of the model (s, g, n, δ) and on the curvature of the production function $(f'(\cdot))$.

Rate of growth of k

- Alternatively we can look at the growth rate of k_t, $g_k = \Delta k_{t+1}/k_t$

$$g_k = \frac{\Delta k_{t+1}}{k_t} = \frac{\frac{sf(k_t)+(1-\delta)k_t}{(1+g)(1+n)} - k_t}{k_t} = \frac{\frac{sf(k_t)}{k_t} - (g+n+\delta+gn)}{(1+g)(1+n)}$$

- As for small value of x, $\ln(1+x) = x$ and $g_k = \frac{k_{t+1}}{k_t} - 1$ we can write $g_k \approx \ln\left(\frac{k_{t+1}}{k_t}\right)$
- Computing first-order Taylor expansion around $\ln k_t = \ln k^*$ yields

$$g_k \approx g_k\Big|_{\ln k_t = \ln k^*} + \frac{s}{(1+g)(1+n)}\left[\frac{f'(k_t)k_t}{k_t} - \frac{f(k_t)k_t}{k_t^2}\right]\Bigg|_{\ln k_t = \ln k^*} (\ln k_t - \ln k^*)$$

$$\approx \frac{s}{(1+g)(1+n)}\frac{f(k^*)}{k^*}(\alpha(k^*) - 1)\ln\left(\frac{k_t}{k^*}\right)$$

$$\approx -\frac{g+n+\delta+g\cdot n}{(1+g)(1+n)}(1-\alpha(k^*))\ln\left(\frac{k_t}{k^*}\right)$$

where we used $k_t = e^{\ln k_t}$ and $\left(e^k\right)' = e^k$.

- The speed of convergence depends negatively on $\alpha(k^*)$.

Convergence

- We can expressed this equation as

$$\ln\left(\frac{k_{t+1}}{k_t}\right) = -\omega \ln\left(\frac{k_t}{k^*}\right)$$

where

$$\omega = \frac{g+n+\delta+g\cdot n}{(1+g)(1+n)}\left(1-\alpha(k^*)\right)$$

- Noting that

$$\ln\left(\frac{k_{t+1}}{k_t}\right) = \ln k_{t+1} - \ln k_t - \ln k^* + \ln k^* = \ln\left(\frac{k_{t+1}}{k^*}\right) - \ln\left(\frac{k_t}{k^*}\right)$$

we get

$$\ln\left(\frac{k_{t+1}}{k^*}\right) = (1-\omega)\ln\left(\frac{k_t}{k^*}\right)$$

$\Longrightarrow$ each period $\omega\%$ of the gap between capital (or output) and its steady-state level disappears.

Implications

- Important use of the Solow model is that it allows us to look at the data at two different levels.

  1. Growth over time
  2. Income level differences

How large are the contributions of capital and labor to output growth?

- Growth accounting provides an accounting answer.
- Consider a production function with Hicks neutral technological progress
$$Y = A_t F(K_t, L_t)$$
- Growth rate of GDP per worker with given the production function:

$$\begin{aligned} g_{Y/L} &= g_Y - g_L \\ &= \alpha(K) g_{K/L} + g_A \end{aligned}$$

Estimating the variables: capital stock

- Estimate from past accumulated investment (perpetual inventory method).
- Start from an arbitrary K_0 way back in the past.
- Using data on investment, I_t, compute

$$K_{t+1} = (1 - \delta)K_t + I_t$$

  by forward iteration.
- Quality adjustments can be made to account for the fact that newer vintages of K are more productive.

Estimating the variables: labor input

- Estimating the variables: capital stock

- Quality adjustment can be made to account for the fact that more educated workers are more productive (etc.).

Factor income shares

- Recall that capital receives fraction α of total income

$$\alpha = \frac{rK}{Y}$$

- While labor receives $1 - \alpha$

$$1 - \alpha = \frac{wL}{Y}$$

Total Factor Productivity (TFP)

- A is unobservable.
- It can be computed as a residual from the equation

$$g_A = g_Y - \alpha g_K - (1 - \alpha) g_L$$

Empirical results: OECD countries

Table 10.1
Growth Accounting for a Sample of Countries

Country	(1) Growth Rate of GDP	(2) Contribution from Capital	(3) Contribution from Labor	(4) TFP Growth Rate
Panel A: OECD Countries, 1947–73				
Canada	0.0517	0.0254	0.0088	0.0175
($\alpha = 0.44$)		(49%)	(17%)	(34%)
France[a]	0.0542	0.0225	0.0021	0.0296
($\alpha = 0.40$)		(42%)	(4%)	(54%)
Germany[b]	0.0661	0.0269	0.0018	0.0374
($\alpha = 0.39$)		(41%)	(3%)	(56%)
Italy[b]	0.0527	0.0180	0.0011	0.0337
($\alpha = 0.39$)		(34%)	(2%)	(64%)
Japan[b]	0.0951	0.0328	0.0221	0.0402
($\alpha = 0.39$)		(35%)	(23%)	(42%)
Netherlands[c]	0.0536	0.0247	0.0042	0.0248
($\alpha = 0.45$)		(46%)	(8%)	(46%)
U.K.[d]	0.0373	0.0176	0.0003	0.0193
($\alpha = 0.38$)		(47%)	(1%)	(52%)
U.S.	0.0402	0.0171	0.0095	0.0135
($\alpha = 0.40$)		(43%)	(24%)	(34%)

- Close to half of the growth is due to productivity.
- The rest is mostly due to capital accumulation.
- Note the productivity slowdown after 1973.

Empirical results: OECD countries

Table 10.1
Growth Accounting for a Sample of Countries

Country	(1) Growth Rate of GDP	(2) Contribution from Capital	(3) Contribution from Labor	(4) TFP Growth Rate
Panel B: OECD Countries, 1960–95				
Canada	0.0369	0.0186	0.0123	0.0057
($\alpha = 0.42$)		(51%)	(33%)	(16%)
France	0.0358	0.0180	0.0033	0.0130
($\alpha = 0.41$)		(53%)	(10%)	(38%)
Germany	0.0312	0.0177	0.0014	0.0132
($\alpha = 0.39$)		(56%)	(4%)	(42%)
Italy	0.0357	0.0182	0.0035	0.0153
($\alpha = 0.34$)		(51%)	(9%)	(42%)
Japan	0.0566	0.0178	0.0125	0.0265
($\alpha = 0.43$)		(31%)	(22%)	(47%)
U.K.	0.0221	0.0124	0.0017	0.0080
($\alpha = 0.37$)		(56%)	(8%)	(36%)
U.S.	0.0318	0.0117	0.0127	0.0076
($\alpha = 0.39$)		(37%)	(40%)	(24%)

- Close to half of the growth is due to productivity.
- The rest is mostly due to capital accumulation.
- Note the productivity slowdown after 1973.

Empirical results: Other countries

- The Newly Industrialized Countries (NICs) grew at spectacular rates.

Table 10.3
TFP Growth Adjusted for Endogenous Responses of Capital

Country	(1) GDP Growth Rate	(2) TFP Growth Rate	(3) TFP Growth Adjusted for Physical Capital	(4) TFP Growth Adjusted for Broad Capital
Hong Kong	0.073	0.027 (37%)	0.043 (59%)	0.090 (123%)
Singapore	0.087	0.022 (25%)	0.043 (49%)	0.073 (84%)
South Korea	0.103	0.015 (14%)	0.021 (20%)	0.050 (49%)
Taiwan	0.094	0.037 (39%)	0.050 (53%)	0.123 (131%)

Notes: Column 1 shows the growth rate of GDP as given in table 10.1, panel D. Column 2 shows the TFP growth rate indicated for the dual column in table 10.2. Column 3 adjusts for responses of physical capital by multiplying the TFP growth rate by $1/(1-\alpha)$, where α is the capital share shown in table 10.1, panel D. Column 4 adjusts for responses of physical and human capital by multiplying the TFP growth rate by $1/0.3$, that is, by assuming a broad capital share of $\alpha = 0.7$. The numbers in parentheses show the percentages of the growth rate of GDP accounted for by each measure of TFP growth.

- Hong Kong had an average growth of 7.3% but only 37% of that is due to TFP growth.

Alwyn Young (1995)

- Take a labor augmenting (Harrod-neutral) production function

$$Y_t = K_t^{\alpha}(A_t L_t)^{1-\alpha}$$

- Rate of growth can be written as

$$g_Y = \alpha g_K + (1-\alpha) g_L + (1-\alpha) g_A$$

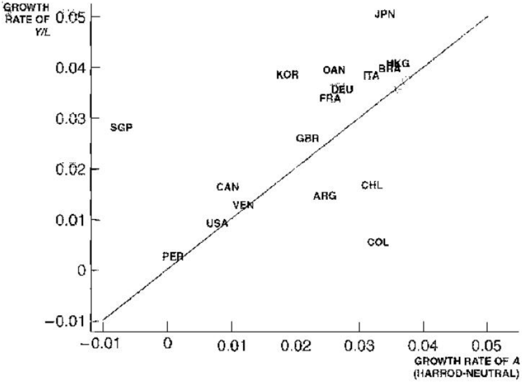
- For some countries it is easier to measure K/Y ratio.
- Rewrite above equation as

$$\alpha g_Y + (1-\alpha) g_Y = \alpha g_K + (1-\alpha) g_L + (1-\alpha) g_A$$

- The growth rate of per-capita output

$$g_{Y/L} = \frac{\alpha}{1-\alpha} g_{K/Y} + g_A$$

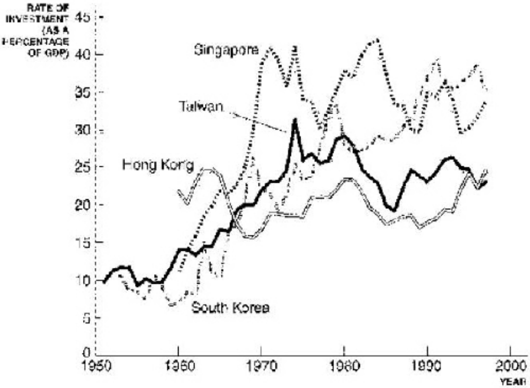
TFP growth around the world



Barro and Sala-i-Martin (1998). [OECD 1960-90, LA 1940-80, EA 1966-90]

- The Asian Tigers do not have particularly fast TFP growth.
- TFP growth in SGP is negative! OAN(Taiwan) is below COL(Columbia).

How did the NICs sustain high growth?



- The key is high and rising investment.

Is rapid growth sustainable?

- Recall

$$g_{Y/L} = (1-\alpha)g_A + \alpha g_{K/L}$$

- In the long-run, K/L and Y/L must grow at the same rate

$$g_{Y/L} = (1-\alpha)g_A + \alpha g_{Y/L}$$

- Long-run growth is therefore entirely determined by TFP growth:

$$g_{Y/L} = g_A$$

- Temporarily, countries can sustain faster growth by raising I/Y .
- But when I/Y levels off, growth drops to the level determined by g_A.

Summing-up

- Growth accounting quantifies the contributions of K and A to GDP growth.

- It answers the question: How much would GDP growth have changed, had K/L or A been constant over time?
  - For most countries: the share of growth due to TFP is large (2/3 in the U.S.).
  - Temporarily, growth can deviate from g_A through rising I/Y .
  - But in the long-run, growth is entirely dictated by g_A.

Limitations

- Growth accounting does not identify sources of growth

- At least in the Solow model: all y growth is ultimately due to technical change.

- But growth accounting attributes fraction α of g_y to g_k.

Level accounting

- Cross-country income gaps:
  - In the data: Richest countries produce around 30 times more output per worker than poorest countries.
  - Can the Solow model account for these differences?
  - How important is capital vs. productivity (A)?

Physical Capital

- Consider a country has $1/10$ of US per capita income.
- Suppose we assume that every country has the same level of productivity (A)
- Assume the Cobb-Douglas production function from the Solow model:

$$y = A^{1-\alpha}k^{\alpha}$$

where $y = Y/L$ and $k = K/L$.

- Then

$$\frac{y_i}{y_{US}} = \left(\frac{k_i}{k_{US}}\right)^{\alpha}$$

- In the data $\alpha = 1/3$ so if $\frac{y_i}{y_{US}} = \frac{1}{10}$ then

$$\frac{k_i}{k_{US}} = \left(\frac{y_i}{y_{US}}\right)^{1/\alpha} = \left(\frac{1}{10}\right)^3 = \frac{1}{1000}$$

- Rates of return to capital (before depreciation):

$$r_i = \alpha A^{1-\alpha} k_i^{\alpha-1}$$

- Compare rich and poor countries:

$$\frac{r_i}{r_{US}} = \left(\frac{k_i}{k_{US}}\right)^{\frac{1}{3}-1} = \left(\frac{k_{US}}{k_i}\right)^{\frac{2}{3}} = 1000^{2/3} = 100$$

- Real interest rate in the poorer country will have to be higher than U.S. by a factor of 100.
- But poor countries have higher interest rates but the order of magnitude is roughly 1.8.

- What if value of $\alpha = 0.8$?
- Consider again a poor country with $1/10$ of U.S. y.
- Repeat the calculations from the Solow model with high α

$$\frac{k_i}{k_{US}} = \left(\frac{y_i}{y_{US}}\right)^{1/\alpha} = \left(\frac{1}{10}\right)^{1.25} = \frac{1}{17}$$

- The implied interest rate differential

$$\frac{r_i}{r_{US}} = \left(\frac{k_i}{k_{US}}\right)^{\frac{0.8}{-}1} = 17^{0.2} = 1.8$$

- But the share of capital income in GDP is much less than 0.8.

- What can we infer from this?
  - Countries are not poor simply because they lack physical capital.
  - Accounting for cross-country income gaps on the order of 10 requires differences in TFP (A).
  - A main goal of growth research is to identify what is "inside" the TFP residual.

Generalizing the Solow Model

- Mankiw, Romer, and Weil (1992) propose a solution to the quantitative problems of the Solow model.
- The Solow model has only one type of capital which an economy accumulates. Perhaps there are other kinds of capital which are equally if not more important than physical capital (this is also a subject of study of endogenous growth theory).
- What is the missing capital? Possible candidates: human capital (education, experience), organization capital, etc.

A model with human capital

- Output is produced from physical and human capital:

$$Y_t = K_t^{\alpha}(A_t H_t)^{1-\alpha}$$

- Physical capital is accumulated as usual:

$$\Delta K_t = sY_t - \delta K_t$$

- Human capital (think schooling) augments the productivity of labor:

$$H_t = h_t L_t$$

- Hence we can write output or income per capita for any country i as (note that $y = Y/L$ and $k = K/L$):

$$y_t = k_t^{\alpha}(A_t h_t)^{1-\alpha}$$

The Solow Residual

- In the model: A differences account for the remaining income gaps.
- How to measure A?
- Write the production function as

$$
\begin{aligned}
y_i &= k_i^{\alpha} (A_i h_i)^{1-\alpha} \\
&= \left(\frac{k_i}{y_i} \right)^{\frac{\alpha}{1-\alpha}} A_i h_i
\end{aligned}
$$

- Estimate A as a residual:

$$
A_i = \frac{y_i}{h_i} \left(\frac{K_i}{Y_i} \right)^{-\frac{\alpha}{(1-\alpha)}}
$$

- This is called the Solow residual.

What does A represent?

- Everything other than K and H that affects productivity
- Technology, institutions,...

[...] "A measure of our ignorance"

Abramovitz, M. (1956) "Resource and Output Trends in the United States since 1870," *American Economic Review* 46: 5-23.

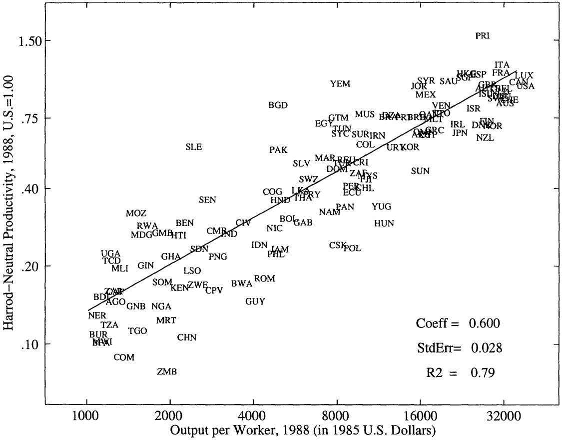
- Goal of research on growth:
  - Identify what is "inside" the Solow residual.
  - Measure as much of it as we can. Shrink the residual towards 0.

How important is TFP?

- To answer this question: collect data on y, h, k/y and decompose income gaps into 3 contributions:

$$\frac{y_i}{y_{US}} = \frac{A_i}{A_{US}} \frac{h_i}{h_{US}} \left(\frac{k_i/y_i}{k_{US}/y_{US}} \right)^{\frac{\alpha}{1-\alpha}}$$

Empirical results, Hall and Jones (QJE 1999)



Hall and Jones (1999), Figure 1

- Figure shows A_i against y_i for 1997.
- Very high correlation (0.89): rich countries have high TFP.

Contributions to cross-country income gaps

PRODUCTIVITY CALCULATIONS: RATIOS TO U. S. VALUES

Country	Y/L	Contribution from $(K/Y)^{\alpha/(1-\alpha)}$	H/L	A
United States	1.000	1.000	1.000	1.000
Canada	0.941	1.002	0.908	1.034
Italy	0.834	1.063	0.650	1.207
West Germany	0.818	1.118	0.802	0.912
France	0.818	1.091	0.666	1.126
United Kingdom	0.727	0.891	0.808	1.011
Hong Kong	0.608	0.741	0.735	1.115
Singapore	0.606	1.031	0.545	1.078
Japan	0.587	1.119	0.797	0.658
Mexico	0.433	0.868	0.538	0.926
Argentina	0.418	0.953	0.676	0.648
U.S.S.R.	0.417	1.231	0.724	0.468
India	0.086	0.709	0.454	0.267
China	0.060	0.891	0.632	0.106
Kenya	0.056	0.747	0.457	0.165
Zaire	0.033	0.499	0.408	0.160
Average, 127 countries:	0.296	0.853	0.565	0.516
Standard deviation:	0.268	0.234	0.168	0.325
Correlation with Y/L (logs)	1.000	0.624	0.798	0.889
Correlation with A (logs)	0.889	0.248	0.522	1.000

The elements of this table are the empirical counterparts to the components of equation (3), all measured as ratios to the U. S. values. That is, the first column of data is the product of the other three columns.

Hall and Jones (1999), Table 1

Main findings

- For low income countries, TFP is by far the largest contributor to income gaps.
- Overall, TFP accounts for about $2/3$ of cross-country income gaps.
- Example
  - Poorest 5 countries with $y_i/y_{US} = 1/32$.
  - Contributions are:
    - Capital: 1.8
    - Education: 2.2
    - TFP: 8.3
- Conclusion: Capital and Human Capital differences can only partially explain the cross-country income differences.

Limitations

- This is an accounting exercise
  - It does not tell us about the causes of cross-country income gaps.
  - But it provides indirect information about the causes: they must work primarily through TFP.

- Questions:
  - Is years of schooling likely a good measure of human capital?
  - We don't know what TFP represents: technology, institutions,...
  - Levels accounting is mainly useful as a guide for constructing and evaluating theories.