Lecture 8: Overlapping Generation Model

Advanced Macroeconomics

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Introduction

- OLG models are used in variety of applications.

- They have features that make them distinct:
  - competitive equilibria need not be Pareto Optimal,
  - Ricardian equivalence does not hold.

  $\Rightarrow$ There is a role for government intervention.

- Agents in the model are heterogeneous with respect to
  - age,
  - (possibly) individual productivity.

Basic OLG model

- Time is discrete and continues forever: $t = 0, 1, 2 \dots$
- In each period, $L_t \in \mathbb{R}^+$ two-period-lived agents are born.
- An agent is endowed with one unit of labor in the first period and with zero units of labor in the second period.

  (agents can work only when young)
- Population evolves according to $L_t = (1+n)^t L_0$.

  (if $n = 0$ no population growth)

Agents

- At date 0 there are some *old* agents who live for one period and are collectively endowed with $K_0 \in \mathbb{R}^+$ units of capital.
- An agent born at date t has preferences represented by

$$u(c_t^y, c_{t+1}^o).$$

- c_t^y denotes the consumption of a young agent at date t.
- c_t^o denotes the consumption of an old agent at date t.
- $u(\cdot)$ is utility function that is strictly increasing in both arguments, strictly concave, and continuously differentiable.

Utility

- Define the ratio

$$v(c^y, c^o) \equiv \frac{u_1(c^y, c^o)}{u_2(c^y, c^o)}.$$

- Then

$$\lim_{c^y \to 0} v(c^y, c^o) = \infty, \qquad \text{for any } c^o > 0,$$

$$\lim_{c^o \to 0} v(c^y, c^o) = 0, \qquad \text{for any } c^y > 0.$$

- The initial old seek to maximize consumption at date 0.

Investment technology and capital

- Consumption goods can be converted one-for-one into capital and vice versa.
- Capital constructed at date t becomes productive only at date $t + 1$.
- There is no capital depreciation ($\delta = 0$).
- Young agents sell their labor to firms and save in the form of capital accumulation.
- Old agents rent capital to firms.
- After production takes place, old agents convert capital into goods for consumption.

Firms

- Representative firm rents capital and hires labor to produce goods.

- Technology is represented by the production function

$$Y_t = F(K_t, L_t),$$

  - Y_t represents output,
  - K_t denotes capital input,
  - L_t denotes labor input,
  - $F(\cdot,\cdot)$ is strictly increasing in each argument, strictly quasiconcave, twice differentiable, and homogeneous of degree one.

Young Consumer's Problem

- An agent born at date t solves the problem

$$\max_{c_t^y, c_{t+1}^o, s_t} u(c_t^y, c_{t+1}^o)$$

subject the budget constraints

$$c_t^y + s_t \leq w_t,$$

$$c_{t+1}^o \leq (1 + r_{t+1}) s_t,$$

  - s_t is saving when young,
  - w_t is the wage rate at date t,
  - r_{t+1} is the capital rental rate at date $t + 1$.
- Consumer chooses savings and consumption when young and old treating prices, w_t and r_{t+1}, as being fixed.
- Agents have perfect foresight (i.e., they can perfectly predict prices): at time t the consumer is assumed to know r_{t+1}.

- Substituting for c_t^y and c_{t+1}^o from budget constraints reduces the maximization problem to the choice of one variable (s_t):

$$\max_{s_t} u(w_t - s_t, (1 + r_{t+1})s_t).$$

- The first-order condition for an optimum is

$$-u_1(w_t - s_t, (1 + r_{t+1})s_t) + u_2(w_t - s_t, (1 + r_{t+1})s_t)(1 + r_{t+1}) = 0.$$

- It determines the optimal savings, s_t, as a function of prices

$$s_t = s(w_t, r_{t+1}).$$

- We can also write the FOC as

$$\frac{u_1}{u_2} = 1 + r_{t+1}$$

(the intertemporal marginal rate of substitution equals one plus the interest rate)

- If consumption when young and consumption when old are both normal goods, then $\frac{\partial s}{\partial w_t} > 0$.
- The same does not have to be true for $\frac{\partial s}{\partial r_{t+1}}$ due to opposing income and substitution effects.

Representative Firm's Problem

- Representative firm solves a one period, static problem

$$\max_{K_t, L_t} \left[F(K_t, L_t) - w_t L_t - r_t K_t \right].$$

- The first-order conditions for a maximum profits are

$$\begin{aligned} F_K(K_t, L_t) - r_t &= 0, \\ F_L(K_t, L_t) - w_t &= 0. \end{aligned}$$

- Since the production function is homogenous of degree one, we can define

$$f(k_t) \equiv F(k_t, 1) = F(K_t/L_t, 1),$$

where $k_t = K_t/L_t$ is capital per worker (not per capita) .

- Then,

$$\begin{aligned} r_t &= f'(k_t), \\ w_t &= f(k_t) - k_t f'(k_t). \end{aligned}$$

Definition: Competitive Equilibrium

Definition

A *competitive equilibrium* is a sequence of quantities $\{k_{t+1}, s_t\}_{t=0}^{\infty}$ and prices $\{w_t, r_t\}_{t=0}^{\infty}$ which satisfy

- consumer's optimization, ($s_t = s(w_t; r_{t+1})$);
- firm's optimization, ($r_t = f'(k_t)$ and $w_t = f(k_t) - k_t f'(k_t)$); and
- market clearing, ($(1+n)k_{t+1} = s(w_t, r_{t+1})$)

at any date $t \geq 0$ given the initial capital-labor ratio k_0.

Characterization of Equilibrium

- Market-clearing condition for the capital rental market is given by

$$K_{t+1} = L_t s(w_t, r_{t+1}).$$

- We can rewrite it as

$$\frac{L_{t+1}}{L_t}\frac{K_{t+1}}{L_{t+1}} = s(w_t, r_{t+1}) \quad \Rightarrow \quad (1+n)k_{t+1} = s(w_t, r_{t+1}).$$

- As $r_t = f'(k_t)$ and $w_t = f(k_t) - k_t f'(k_t)$ it can be written as

$$(1+n)k_{t+1} = s(f(k_t) - k_t f'(k_t), f'(k_{t+1})).$$

- Given k_0, the equilibrium evolution of the capital-labor ratio is given by

$$(1+n)k_{t+1} = s(f(k_t) - k_t f'(k_t), f'(k_{t+1})).$$

- It is a nonlinear first-order difference equation (solution $k_{t+1} = g(k_t)$).

- Given the equilibrium sequence of the capital-labor ratio, we can recover the equilibrium prices from

$$r_t = f'(k_t),$$
$$w_t = f(k_t) - k_t f'(k_t).$$

Figure 1: Dynamic System

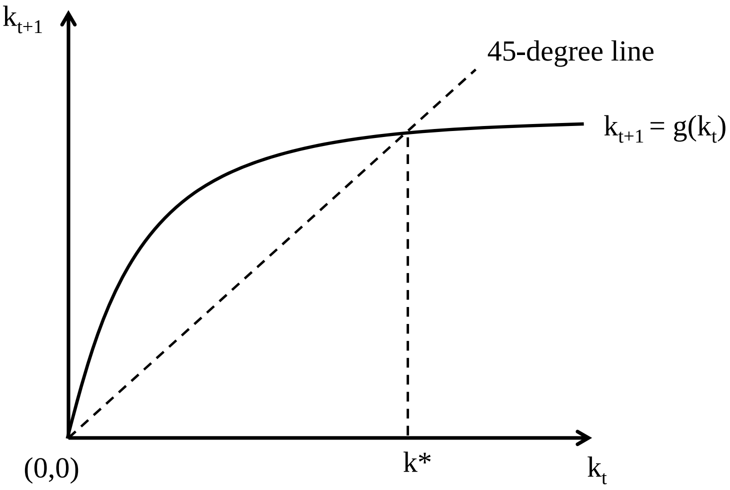
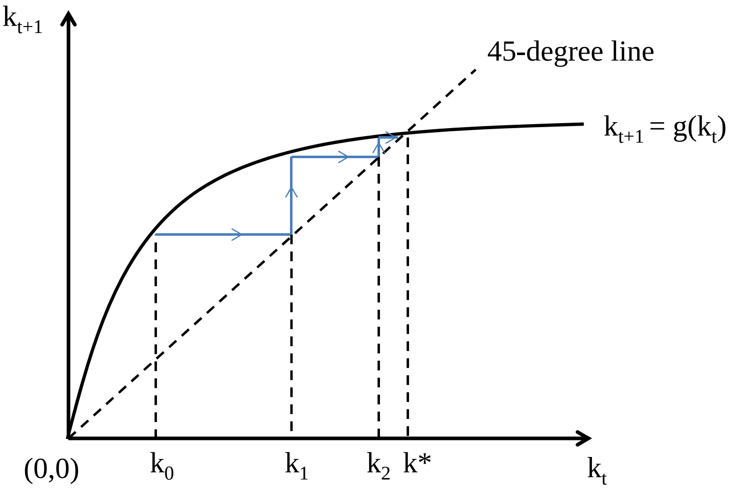


Figure 1: Dynamic System



Definition: Optimal Allocation

Definition

A *Pareto optimal allocation* is a sequence $\{c_t^y, c_t^o, k_{t+1}\}_{t=0}^{\infty}$ satisfying resource constraint

$$f(k_t) + k_t = (1+n)k_{t+1} + c_t^y + \frac{c_t^o}{1+n}$$

$$\left(\quad \text{or} \qquad F(K_t, L_t) + K_t = K_{t+1} + c_t^y L_t + c_t^o L_{t-1}, \qquad \right)$$

and with the property that there exists no other feasible allocation $\{\hat{c}_t^y, \hat{c}_t^o, \hat{k}_{t+1}\}_{t=0}^{\infty}$ satifying

$$\hat{c}_t^o \geq c_t^o$$
$$u(\hat{c}_t^y, \hat{c}_{t_1}^o) \geq u(c_t^y, c_{t_1}^o)$$

for all $t \geq 0$, with strict inequality in at least one instance.

Optimal Allocation

- A Pareto optimal allocation is a feasible allocation such that there is no other feasible allocation for which all consumers are at least as well off and some consumer is better off.

- To characterize Pareto optimal allocations solve the social planner's problem who has control over production, capital accumulation, and the distribution of consumption goods between the young and the old.

- Solve the planner's problem to characterize Pareto optimal allocations.

- For simplicity, suppose $u(c_t^y, c_{t+1}^o) = h(c_t^y) + \beta h(c_{t+1}^o)$
  h is increasing, strictly concave, and continuously differentiable.

- Suppose, for now, that the planner cares only about the utility of the current and T future generations.

Social Planner Problem

- Define the social welfare function

$$W_T = \beta h(c_0^o) + \sum_{t=0}^{T-1} \left[h(c_t^y) + \beta h(c_{t+1}^o) \right]$$

- Planner chooses $\{c_0^o, c_0^y, k_1, c_1^o, c_1^y, k_2, \ldots, c_{T-1}^o, c_{T-1}^y, k_T, c_T^o\}$ to maximize social welfare subject to resource constraint

$$f(k_t) + k_t = (1+n)k_{t+1} + c_t^y + \frac{c_t^o}{1+n}$$

  and terminal condition for k_{T+1}.
- After substituting c_t^y from the resource constraint, the FOCs to the social planner problem are

$$\partial W_T / \partial c_t^o : \qquad \beta h'(c_t^o) - \frac{h'(c_t^y)}{1+n} = 0$$

$$\partial W_T / \partial k_{t+1} : \qquad -(1+n)h'(c_t^y) + [f'(k_{t+1}) + 1]h'(c_{t+1}^y) = 0$$

  for all $t = 0, \ldots, T-1$.

Stationary Solution

- Consider a stationary solution to the planner's problem:

$$c_t^o = c^o, \quad c_t^y = c^y, \quad k_{t+1} = k \qquad \text{for all } t = 0, 1, \ldots, T-1.$$

- A stationary allocation (c^y, c^o, k) satisfies

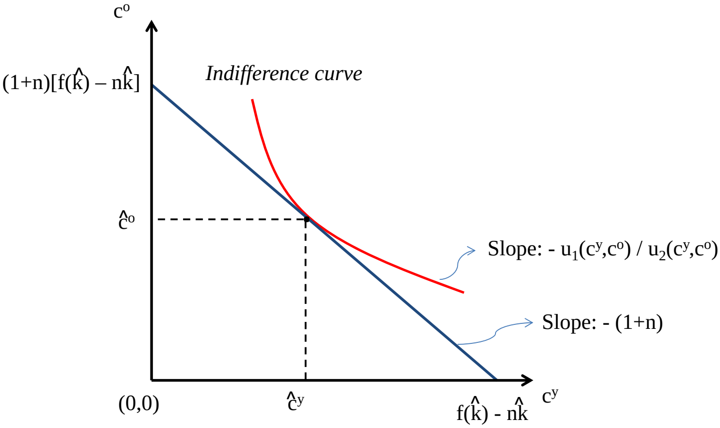
$$(1+n)h'(c^o) = h'(c^y),$$
$$f'(k) = n$$

and

$$f(k) - nk = c^y + \frac{c^o}{1+n}$$

- Stationary solution remains the same for $T \to \infty$.

Figure 2: Pareto Optimum



Example

- Suppose $h(c) = \ln(c)$ and $F(K, L) = \gamma K^{\alpha} L^{1-\alpha}$ with $0 < \alpha < 1$ and $\gamma > 0$.
- Young agent solves

$$\max_{s_t} \left[\ln(w_t - s_t) + \beta \ln([1 + r_{t+1}] s_t) \right],$$

which yields an optimal saving function

$$s_t = \frac{\beta}{1 + \beta} w_t$$

- The FOCs for the firm's profit maximization problem give

$$r_t = \gamma \alpha k_t^{\alpha - 1}$$
$$w_t = \gamma (1 - \alpha) k_t^{\alpha}$$

k^*

- The FOCs for a firm and young agent problem, together with $k_{t+1}(1+n) = s_t$, give equilibrium law of motion for capital accumulation
$$k_{t+1}(1+n) = \frac{\beta}{1-\beta}\gamma(1-\alpha)k_t^{\alpha}.$$

- In this case, for a given k_0 we obtain a unique equilibrium sequence for capital $\{k_t\}_{t=0}^{\infty}$.

- In the competitive equilibrium, the steady state is given by
$$k^* = \left[\frac{\beta}{1-\beta}\frac{\gamma(1-\alpha)}{1+n}\right]^{\frac{1}{1-\alpha}}.$$

$\hat{k}$

- We can compute the steady-state for the planner problem.
- For our functional assumptions, the social planner allocation satisfied

$$(1+n)h'(c^o) = h'(c^y),$$
$$f'(k) = n,$$

which, for our functional assumptions, equals

$$(1+n)c^y = c^o,$$
$$\gamma\alpha k^{1-\alpha} = n.$$

- Socially optimal steady state capital stock is

$$\hat{k} = \left[\frac{\gamma\alpha}{n}\right]^{\frac{1}{1-\alpha}}.$$

Dynamic inefficiency

- In general, $k^* \neq \hat{k}$:

  $\Rightarrow$ the competitive equilibrium steady state is in general not socially optimal

- Economy suffers from a dynamic inefficiency:

  depending on parameters, there might be too much or too little capital in the steady.

- Suppose $\beta = 1$ and $n = 0.3$:
  - if $0 < \alpha < 0.103$ then $k^* > \hat{k}$, and
  - if $0.103 < \alpha < 1$ then $\hat{k} > k^*$.

- Inefficiency due to $k^* \neq \hat{k}$:
  1. quantity of resources available to allocate between the young and the old is not optimal,
  2. $r^* \neq n$ so consumers misallocate consumption goods over time:
     $\Rightarrow$ too much or too little saving in a competitive equilibrium .

Government Debt

- Government debt can used to introduce intergenerational transfers.

- Government issues debt to young agents, transfers the proceeds to old agents, and then taxes the young of the next generation in order to pay the interest and principal on the debt.

- B_{t+1} is the quantity of one-period bonds issued by the government at date t (and bought backed in period $t+1$).
- Bond is a promise to pay $1 + r_{t+1}$ units of consumption goods at date $t + 1$.
- Interest rate on government bonds is the same as the rental rate of capital:
  - government bonds and capital are perfect substitutes in equilibrium .
- Assume $B_{t+1} = bL_t$ with $b > 0$ fixed: the quantity of government debt is fixed in per-capita terms.

Taxes

- Quantity of government debt is fixed in per-capita terms.
- Taxes T_t are levied lump-sum on young agents.
- Government's budget constraint is

$$B_{t+1} + T_t = (1 + r_t)B_t,$$

i.e. the revenues from new bond issues and taxes in period t, T_t, equals the payments of interest and principal on government bonds issued in period $t - 1$.

- Let τ_t denote the lump-sum tax per young agent

$$T_t = \tau_t L_t.$$

- Solving for τ_t yields

$$\tau_t = \frac{T_t}{L_t} = \frac{(1 + r_t)B_t - B_{t+1}}{L_t} = \left(\frac{r_t - n}{1 + n}\right) b.$$

Young Consumer's Problem

- Young agent solves

$$\max_{s_t} u(w_t - s_t \tau_t, (1 + r_{t+1}) s_t).$$

- Agents save in the form of capital and government bonds.
- Optimal savings for a young agent is given by

$$s_t = s(w_t - \tau_t, r_{t+1}).$$

- As before, profit maximization by the firm implies

$$r_t = f'(k_t),$$
$$w_t = f(k_t) - k_t f'(k_t).$$

Definition: Competitive Equilibrium with Government Debt

Definition

A *competitive equilibrium* is a sequence of quantities $\{k_{t+1}, s_t, \tau_t\}_{t=0}^{\infty}$ and prices $\{w_t, r_t\}_{t=0}^{\infty}$ which satisfy

- consumer's optimization, $(s_t = s(w_t - \tau_t, r_{t+1}))$;
- firm's optimization, $(r_t = f'(k_t)$ and $w_t = f(k_t) - k_t f'(k_t))$;
- market clearing, $((1+n)k_{t+1} + b = s(w_t - \tau_t, r_{t+1}))$; and
- government budget constraint, $\left(\tau_t = \left(\frac{r_t - n}{1+n}\right) b\right)$,

at any date $t \geq 0$ given the initial capital-labor ratio k_0.

Characterization of Equilibrium with Government Debt

- Using market cleaning condition, equilibrium law of motion for the capital-labor ratio is given by

$$(1+n)k_{t+1} + b = s\left(f(k_t) - k_t f'(k_t) - \left(\frac{f'(k_t) - n}{1+n}\right) b, f'(k_{t+1})\right) \quad .$$

- As before, given k_0, we obtain a unique equilibrium sequence $\{k_t\}_{t=0}^{\infty}$.
- Steady state capital-labor ratio k^* satisfies therefore

$$(1+n)k^* + b = s\left(f(k^*) - k^* f'(k^*) - \left(\frac{f'(k^*) - n}{1+n}\right) b, f'(k^*)\right) \quad .$$

- Note that we have $k^* = k^*(b)$ (i.e., capital-labor ratio depends on the government's fiscal policy b) .

Optimal level of b

- Since k^* depends on fiscal policy, social planner can use government bond to restore/implement Pareto optimal allocation.
- Choose the fiscal policy b such that the competitive equilibrium steady state is socially optimal.
- In other words, find $\hat{b}$ such that $k^*(\hat{b}) = \hat{k}$.
- Because $f(\hat{k}) = n$, it follows that

$$\hat{b} = -(1+n)\hat{k} + s(f(\hat{k}) - n\hat{k}, n) \quad .$$

- Note that $\hat{b}$ can be positive or negative.

Optimal level of τ_t

- In the steady state with $b = \hat{b}$, $\tau_t = 0$ and, hence, $T_t = 0$.

$\Rightarrow$ Debt increases at a rate just sufficient to pay the interest and principal on previously issued debt.

- Optimum government debt policy transfers wealth:
  - from the young to the old if the debt is positive, or
  - from the old to the young if the debt is negative .

Example

- Consider as before $u(c^y, c^o) = \ln c^y + \beta \ln c^o$ and $F(K, L) = \gamma K^\alpha L^{1-\alpha}$.
- Optimal savings for a young agent is

$$s_t = \left(\frac{\beta}{1+\beta}\right)(w_t - \tau_t).$$

- The equilibrium sequence $\{k_t\}_{t=0}^\infty$ is determined by

$$k_{t+1}(1+n) + b = \left(\frac{\beta}{1+\beta}\right)\left[(1-\alpha)\gamma k_t^\alpha - \frac{(\alpha\gamma k_t^{1-\alpha} - n)b}{1+n}\right].$$

- The steady-state capital-labor ratio $k^*(b)$ solves the equation above with $k^*(b) = k_t = k_{t+1}$.
- The level of debt that brings Pareto-optimal level of steady-state capital-labor ratio is given by

$$\begin{aligned}\hat{b} &= \left(\frac{\beta}{1+\beta}\right)(1-\alpha)\gamma\left(\frac{\alpha\gamma}{n}\right)^{\frac{\alpha}{1-\alpha}} - \left(\frac{\alpha\gamma}{n}\right)^{\frac{1}{1-\alpha}}(1+n) \\ &= \gamma\left(\frac{\alpha\gamma}{n}\right)^{\frac{\alpha}{1-\alpha}}\left[\frac{\beta(1-\alpha)}{1+\beta} - \frac{\alpha}{n}\right].\end{aligned}$$

Ricardian equivalence

- Note that government debt matters.
- Ricardian equivalence does not hold here because the taxes required to pay off the currently-issued debt are not levied on the agents who receive the current tax benefits from a higher level of debt today.
- Here, government debt policy implements the intergenerational transfers that are required to achieve optimality.