Lecture 6: More Notes on Dynamic Programming

The purpose of these notes is to provide some theoretical underpinnings for dynamic programming. See also Adda and Cooper (2002) or Ljunqvist and Sargent (2002). However, more formal presentation is provided in Stokey and Lucas (with Prescott) (1989) - for more mathematical rigorous approach see these texts and their many references.

Recall that we want to solve

$$\max_{\{c_t,k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t U(c_t)$$

subject to

$$c_t = f(k_t) + (1-\delta)k_t - k_{t+1}$$

with k_0 given. Equivalently, the problem that we want to address is the selection of a sequence $\{k_{t+1}\}_{t=0}^{\infty}$ that solves

$$\max_{\{k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t U(f(k_t) + (1-\delta)k_t - k_{t+1})$$

with k_0 given. Here f is a production function assumed to satisfy the standard Inada conditions. Note that the sum is converging as $0 < \beta < 1$ and k is bounded at every t.

1 Non-stochastic case

Consider the more general problem. We want to solve the infinite horizon optimization problem with a concave utility/payoff function given by $r(s_t, c_t)$,

$$\sum_{t=0}^{\infty} \beta^t r(s_t, c_t), \tag{1}$$

where β is the discount factor with $0 < \beta < 1$.

s_t is a state vector: a set of variables that agents treat as given at period t. They influence agent's current payoff but are outside of the agent's control within period t.

c_t is the control vector: a set of variables that agent chooses to maximize its optimization problem.

The evolution of state variables, s_t, depends on c_t and is given by the transition equation

$$s_{t+1} = g(s_t, c_t),$$

Note that given the current state and the current control, the state vector for the subsequent period is fully determined.

We assume that $c \in \mathbb{R}^k$ and that $S = \{(s_{t+1}, s_t) : \; s_{t+1} \leq g(s_t, c_t), c_t \in \mathbb{R}^k\}$ is convex and compact ($s \in S$).

Dynamic programming can be used to solve the problem of maximizing (1) subject to transition equation. Let V be the value function defined as

$$V(s_0) = \max_{\{c_s\}_{s=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t r(s_t, c_t), \tag{2}$$

where maximization is subject to $s_{t+1} = g(s_t, c_t)$, with s_0 given, that is $V(s_0)$ is the optimized value of the problem given the initial state.

If the problem is stationary (the optimal choice of the agent depends only on the state s_t and not on the time t of optimization), we can write

$$V(s_t) = \max_{c_t} r(s_t, c_t) + \beta V(s_{t+1}).$$

with $s_{t+1} = g(s_t, c_t)$, or vy dropping the time subscripts and denoting x' as the next period value of x, we can write a the Bellman equation

$$V(s) = \max_{c} r(s, c) + \beta V(s'), \tag{3}$$

subject to transition equation $s' = g(s, c)$.

If there exists a time-invariant policy function, $h(s_t)$, that maps the state variable s into the control c, $c_t = h(s_t)$, then, using the transition equation $s' = g(s, c)$, we can write the Bellman equation as in Stokey and Lucas (1989),

$$V(s) = \max_{s' \in \Gamma(s)} r(s, s') + \beta V(s'), \tag{4}$$

for all $s \in S$.

Alternatively, if the policy function satisfies the right hand side of equation (3), we can write it as

$$V(s) = r(s, h(s)) + \beta V(g(s, h(s)). \tag{5}$$

These are functional equations in $V(s)$ and $h(s)$ (we can also write a policy function as $s' = \phi(s) = g(s, h(s))$).

Now, instead of solving the original problem of finding an infinite sequence of controls, $\{c_t\}_{t=0}^{\infty}$ that maximizes expression (1), we need to jointly solve for the optimal value function $V(s)$ and a policy function $h(s)$ that solve the continuum of maximum problems (4)—one maximum problem for each value of x. Note that while the payoff and transition equations are primitives of the model, the value and policy functions are derived as the solution of the functional equation, (4).

Results on the existence of a solution to the functional equation depends on mathematical structures of primitive functions r and g. One set of sufficient conditions (taken from Adda and Cooper (2002)) is presented below.[1]

Theorem 1. *Assume $r(s, c)$ is real-valued, continuous, concave, and bounded, $0 < \beta < 1$, and the constraint set S generated by g is non-empty, compact, continuous, and convex. Then*

(i) there exists a unique strictly concave value function $V(s)$ that solves (4),

(ii) there exists a unique, time-invariant, and continuous optimal policy function $c_t = h(s_t)$, (or $s_{t+1} = \phi(s_t)$).

Proof: See Stokey and Lucas (1989),[Theorem 4.6 and 4.7].

The proof of existence and uniqueness of V, which also gives rise to the solution techniques, uses the contraction mapping theorem.

Definition 1 (Contraction Mapping)**.** *Let (S, d) be a metric space and $T : S \to S$ be a function mapping S into itself. T is a* ***contraction mapping*** *(with modulus β) if for some $\beta in(0, 1)$, $d(Tx, Ty) \leq \beta d(x, y)$, for all $x, y \in S$.*

[1]See Sargent (1987) and Stokey and Lucas with Prescott (1989) for additional theorems under alternative assumptions about the payoff and transition functions.

Theorem 2 (Contraction Mapping Theorem). *Let (S,d) be a complete metric space and suppose $T : S \to S$ is a contraction mapping with modulus β. Then T has a unique fixed point v in S, and for any $v_0 \in S$,*

$$Tv = v = \lim_{N\to\infty} T^N v_0$$

Define the operator T as

$$T(W)(s) = \max_{s'\in\Gamma(s)} r(s,s') + \beta W(s'),$$

so that T takes a guess on the value function, W, and through the maximization for all s, produces another value function, $T(W)(s)$. Note that any $V(s)$ such that $V(s) = T(V)(s)$ for all $s \in S$ is a solution to (4).

Using two sufficient condition from Blackwell (1965)–monotonicity and discounting–one can show that $T(W)$ is in fact a contraction mapping.

The fact that $T(W)$ is a contraction allows us to take advantage of the contraction mapping theorem above and obtain both the uniqueness of the fixed point (and hence the uniqueness of our value function) as well as method of computing this fixed point by an iteration process using an arbitrary initial condition.

The second property is used extensively as a means of finding the solution to (4). In particular, the solution is approached in the limit as $j \to \infty$ by iterations on

$$V_{j+1}(s) = \max_c \{r(s,c) + \beta V_j(s')\}, \tag{6}$$

subject to $s' = g(s,c)$, s given, starting from any bounded and continuous initial V_0.

To see this, let $V_0(s)$ for all $s \in S$ be an initial guess of the solution to (4). Consider $V_1 = T(V_0)$. If $V_1 = V_0$ for all $s \in S$, then we have the solution. Else, consider $V_2 = T(V_1)$ and continue iterating until $T(V) = V$ so that the functional equation is satisfied. Since $T(V)$ is a contraction, then the $V(s)$ that satisfies (**??**) can be found from the iteration of $T(V_0(s))$ for any initial guess, $V_0(s)$. This procedure is called value function iteration and is a valuable tool for applied analysis of dynamic programming problems.

We also know that off corners the limiting value function V is differentiable with

$$V'(x) = \frac{\partial r}{\partial x}(x, h(x)) + \beta \frac{\partial g}{\partial x}(x, h(x)) V'(g(x, h(x))), \tag{7}$$

which is a version of a Benveniste and Scheinkman (1979) formula. If transition equation does not contain the state s, then $\frac{\partial g}{\partial x} = 0$, and equation (7) become

$$V'(x) = \frac{\partial r}{\partial x}(x, h(x)). \tag{8}$$