Lecture 4: Solving Linear DSGE Models

We have a set of linear expectational equations:

$$AE_t x_{t+1} + Bx_t + Cv_{t+1} = 0. \tag{A}$$

We seek a solution of the form

$$x_{t+1} = Fx_t + Gv_{t+1}. \tag{B}$$

This solution represents the time series behavior of $\{x_t\}$ as a function of $\{v_t\}$, where v_t is a vector of exogenous innovations, or as frequently referenced, structural shocks.

In this note we use a standard timing for capital, that is

$$y_t = z_t k_t^{\alpha} l_t^{1-\alpha}$$

for production function,

$$k_{t+1} = (1-\delta)k_t + i_t$$

for capital accumulation equation,

$$Div_t = y_t - w_t l_t - r_t k_t$$

for profit/dividend function, and

$$c_t + i_t = w_t l_t + r_t k_t + Div_t$$

for the budget constraint.

The linearized equation with these equations are then as follows.

Equation 1:

$$\sigma(E_t \hat{c}_{t+1} - \hat{c}_t) = (1 - \beta(1-\delta))E_t \hat{r}_{t+1} \tag{1}$$

Equation 2:

$$\varphi\hat{l}_t + \sigma\hat{c}_t = \hat{w}_t \tag{2}$$

Equation 3:

$$\hat{k}_{t+1} = (1-\delta)k_t + \frac{i^{ss}}{k^{ss}}\hat{i}_t \tag{3}$$

Equation 4:

$$(1 - \frac{i^{ss}}{y^{ss}})\hat{c}_t + \frac{i^{ss}}{y^{ss}}\hat{i}_t = \hat{y}_t \tag{4}$$

Equation 5:

$$\hat{y}_t = \hat{z}_t + \alpha\hat{k}_t + (1-\alpha)\hat{l}_t \tag{5}$$

Equation 6:

$$\hat{y}_t - \hat{l}_t = \hat{w}_t \tag{6}$$

Equation 7:

$$\hat{y}_t - \hat{k}_t = \hat{r}_t \tag{7}$$

Equation 8:

$$\hat{z}_t = \rho\hat{z}_{t-1} + \epsilon_t \tag{8}$$

Our original system has 8 equations but to we can reduce the system by eliminating some variables/equations.

First, equations (2) and (6) gives

$$\hat{l}_t = \frac{1}{1+\varphi}\hat{y}_t - \frac{\sigma}{1+\varphi}\hat{c}_t$$

Putting equation (5) and expression for $\hat{l}_t$ yields

$$\begin{aligned}
\hat{y}_t &= z_t + \alpha\hat{k}_t + (1-\alpha)\left(\frac{1}{1+\varphi}\hat{y}_t - \frac{\sigma}{1+\varphi}\hat{c}_t\right) \\
\left(1 - \frac{1-\alpha}{1+\varphi}\right)\hat{y}_t &= z_t + \alpha\hat{k}_t - \frac{(1-\alpha)\sigma}{1+\varphi}\hat{c}_t \\
\hat{y}_t &= \frac{1+\varphi}{\alpha+\varphi}z_t + \frac{1+\varphi}{\alpha+\varphi}\alpha\hat{k}_t - \frac{(1-\alpha)\sigma}{\alpha+\varphi}\hat{c}_t
\end{aligned}$$

Taking from equation (4) formula for $\hat{i}_t$ and putting it into equation (3) yields

$$\hat{k}_{t+1} = (1-\delta)k_t + \frac{y^{ss}}{k^{ss}}\hat{y}_t - \frac{c^{ss}}{k^{ss}}\hat{c}_t$$

and using expression for $\hat{y}_t$,

$$\begin{aligned}
\hat{k}_{t+1} - (1-\delta)k_t &= -\frac{c^{ss}}{k^{ss}}\hat{c}_t + \frac{y^{ss}}{k^{ss}}\hat{y}_t \\
\hat{k}_{t+1} - (1-\delta)k_t &= -\frac{c^{ss}}{k^{ss}}\hat{c}_t + \frac{y^{ss}}{k^{ss}}\left(\frac{1+\varphi}{\alpha+\varphi}\hat{z}_t + \frac{1+\varphi}{\alpha+\varphi}\alpha\hat{k}_t - \frac{(1-\alpha)\sigma}{\alpha+\varphi}\hat{c}_t\right)
\end{aligned}$$

Equation (1) can be written as

$$\begin{aligned}
\sigma(E_t\hat{c}_{t+1} - \hat{c}_t) &= \beta r^{ss}E_t\hat{r}_{t+1} \\
\sigma(E_t\hat{c}_{t+1} - \hat{c}_t) &= \beta r^{ss}E_t\hat{y}_{t+1} - \beta r^{ss}E_t\hat{k}_{t+1}
\end{aligned}$$

which, using expression for y_t above, can be written as

$$\sigma(E_t\hat{c}_{t+1} - \hat{c}_t) = -\beta r^{ss}E_t\hat{k}_{t+1} + \beta r^{ss}\frac{1+\varphi}{\alpha+\varphi}E_t\hat{z}_{t+1} + \beta r^{ss}\frac{1+\varphi}{\alpha+\varphi}\alpha E_t\hat{k}_{t+1} - \beta r^{ss}\frac{(1-\alpha)\sigma}{\alpha+\varphi}E_t\hat{c}_{t+1}$$

or

$$-\sigma\hat{c}_t + \sigma\left(1 + \beta r^{ss}\frac{(1-\alpha)}{\alpha+\varphi}\right)E_t\hat{c}_{t+1} = \beta r^{ss}\left(\frac{1+\varphi}{\alpha+\varphi}\alpha - 1\right)E_t\hat{k}_{t+1} + \beta r^{ss}\frac{1+\varphi}{\alpha+\varphi}E_t\hat{z}_{t+1}.$$

Note that together with equation (8)

$$\hat{z}_t = \rho\hat{z}_{t-1} + \epsilon_t$$

we have transform our original set of equations into the system of 3 equations with 3 endogenous variables (c, k, z), and 1 exogenous variable (ϵ).

We can write it as

$$\kappa_{11}\hat{c}_t + \kappa_{12}E_t\hat{k}_{t+1} + \kappa_{13}E_t\hat{z}_{t+1} + \kappa_{14}E_t\hat{c}_{t+1} = 0 \quad (9)$$

$$\kappa_{21}\hat{c}_t + \kappa_{22}\hat{k}_t + \kappa_{23}\hat{z}_t + \kappa_{24}\hat{k}_{t-1} = 0 \quad (10)$$

$$\hat{z}_t + \rho\hat{z}_{t-1} = \epsilon_t \quad (11)$$

where κ_{ij} contain the 'deep' parameters of the model.

Setting $x_t = \{k_t, c_t\}$ and $v_t = z_t$ the model has the form consistent with equation (A):

$$AE_t x_{t+1} + Bx_t + Cv_{t+1} = 0.$$

with A, B, and C being functions of κs.

(Alternatively, we could set $x_t = \{z_t, k_t, c_t\}$ and $v_t = \epsilon_t$ and solve the model.)

1 Blanchard and Kahn's Method

Blanchard and Kahn(1980) developed the method to solve models written as

$$\begin{bmatrix} x_{1t+1} \\ E_t x_{2t+1} \end{bmatrix} = \tilde{A} \begin{bmatrix} x_{1t} \\ x_{2t} \end{bmatrix} + Ef_t, \quad (12)$$

where the model variables have been divided into an $n \times 1$ vector of endogenous predetermined variables x_{1t} (defined as variables for which $E_t x_{1t+1} = x_{1t+1}$), and an $m \times 1$ vector of endogenous non-predetermined variables x_{2t}. The $k \times 1$ vector f_t contains exogenous forcing variables.

For our model to conform to this specification we need to pre-multiply the entire system by A^{-1} so that

$$E_t x_{t+1} = -A^{-1}Bx_t - A^{-1}Cv_{t+1}.$$

which is possible only if A is invertible. Then $\tilde{A} = -A^{-1}B$ and $E = -A^{-1}C$. In $x_t = \{\hat{c}_t, \hat{k}_t\}$, $\hat{k}_t$ is predetermined (given $\hat{k}_t$ and $\hat{i}_t = \hat{y}_t - \hat{c}_t$, $\hat{k}_{t+1}$ is determined as in

(10)); $\hat{c}_t$ is endogenous but not predetermined (as indicated in (9); and $\hat{z}_t$ is an exogenous forcing variable. Thus in the notation of (12), we seek a specification of the model in the form

$$\begin{bmatrix} \hat{k}_{t+1} \\ E_t\hat{c}_{t+1} \end{bmatrix} = \bar{A} \begin{bmatrix} \hat{k}_t \\ \hat{c}_t \end{bmatrix} + E\hat{z}_t. \tag{13}$$

To solve this system, Blanchard and Khan with a Jordan decomposition of $\tilde{A}$:

$$\tilde{A} = \Lambda^{-1}J\Lambda. \tag{14}$$

where J is a diagonal matrix with the eigenvalues of $\tilde{A}$ along its leading diagonal and zeros in the off-diagonal elements, and Λ is a matrix of the corresponding eigenvectors. The eigenvalue are ordered in increasing absolute value in moving from left to right.[1] In particular, J can be written as

$$J = \begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix}, \tag{15}$$

where the eigenvalues in J_1 lie on or within the unit circle (i.e. of modulus less than one), and those in J_2 lie outside of the unit circle. J_2 is said to be unstable or explosive, since J_2^n diverges as n increases.

To proceed, we partition matrices Λ and E

$$\Lambda = \begin{bmatrix} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{21} & \Lambda_{22} \end{bmatrix}, \quad E = \begin{bmatrix} E_1 \\ E_2 \end{bmatrix}, \tag{16}$$

where Λ_{11} is conformable with J_1, etc.

Blanchard-Khan condition

The Blanchard-Khan condition states that if the number of explosive eigenvalues is equal to the number of non-predetermined variables, the system is said to be saddle–path stable and a unique solution to the model exists.

[1] Eigenvalues of a matrix M are obtained from the solution of equations of the form $Me = \lambda e$, where e is an eigenvector and λ the associated eigenvalue.

On the other hand, if the number of explosive eigenvalues (i.e. greater than 1) exceeds the number of non-predetermined variables then no solution exists (and the system is said to be a source). Finally, if the number of explosive eigenvalues is less than the number of non-predetermined variables, then an infinity of solutions exist (and the system is said to be a sink).

If Blanchard-Khan condition holds then under the Jordan decomposition of $\tilde{A}$, (12) yields

$$\begin{bmatrix} x_{1t+1} \\ E_t x_{2t+1} \end{bmatrix} = \Lambda^{-1} J \Lambda \begin{bmatrix} x_{1t} \\ x_{2t} \end{bmatrix} + \begin{bmatrix} E_1 \\ E_2 \end{bmatrix} f_t. \quad (17)$$

Pre-multiplying by Λ we obtain

$$\begin{bmatrix} \bar{\bar{x}}_{1t+1} \\ E_t \bar{x}_{2t+1} \end{bmatrix} = \begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix} \begin{bmatrix} \bar{x}_{1t} \\ \bar{x}_{2t} \end{bmatrix} + \begin{bmatrix} D_1 \\ D_2 \end{bmatrix} f_t. \quad (18)$$

where $\bar{x}_t = \Lambda x_t$ and $D = \Lambda E$, that is

$$\begin{bmatrix} \bar{x}_{1t} \\ \bar{x}_{2t} \end{bmatrix} = \begin{bmatrix} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{21} & \Lambda_{22} \end{bmatrix} \begin{bmatrix} x_{1t} \\ x_{2t} \end{bmatrix} \quad (19)$$

$$\begin{bmatrix} D_1 \\ D_2 \end{bmatrix} = \begin{bmatrix} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{21} & \Lambda_{22} \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \end{bmatrix}. \quad (20)$$

Under this transformation the two sets of equations are now decoupled, so that the non-predetermined variables depend only upon the unstable eigenvalues of $\tilde{A}$ contained in J_2, as expressed in the lower part of (18). In other words, we can write each time $t+1$ variable as solely a function of predetermined variables, exogenous variables and controls at time t.

To find the solution for the non-predetermined variables, we iterate forward lower portion of (18) as follows. Since

$$E_t \bar{x}_{2t+1} = J_2 \bar{x}_{2t} + D_2 f_{2t}$$

we can write $\bar{x}_{2t}$ as

$$\bar{x}_{2t} = J_2^{-1} E_t \bar{x}_{2t+1} - J_2^{-1} D_2 f_{2t}. \tag{21}$$

Iterating it forward

$$\bar{x}_{2t+1} = J_2^{-1} E_{t+1} \bar{x}_{2t+2} - J_2^{-1} D_2 f_{2t+1}, \tag{22}$$

and substituting into (21) yields[2]

$$\bar{x}_{2t} = J_2^{-2} E_t \bar{x}_{2t+2} - J_2^{-2} D_2 E_t f_{2t+1} - J_2^{-1} D_2 f_{2t}. \tag{23}$$

Since J_2 contains eigenvalues that are outside the unit root, $\lim_{n\to\infty} J_2^{-n} = 0$. Therefore, iterating the substitution gives the following expression

$$\bar{x}_{2t} = -\sum_{i=0}^{\infty} J_2^{-(i+1)} D_2 E_t f_{2t+i}. \tag{24}$$

Using equation (19),

$$\left(\quad \bar{x}_{2t} = \Lambda_{21} x_{1t} + \Lambda_{22} x_{2t}, \quad \right)$$

we can map it back to x_{2t} as

$$x_{2t} = -\Lambda_{22}^{-1} \Lambda_{21} x_{1t} - \Lambda_{22} \sum_{i=0}^{\infty} J_2^{-(i+1)} D_2 E_t f_{2t+i}. \tag{25}$$

In the case of or model, $E_t(f_{2t+i}) = \rho^i \hat{z}_t$, and thus (25) becomes

$$x_{2t} = -\Lambda_{22}^{-1} \Lambda_{21} x_{1t} - \Lambda_{22} J_2^{-1} (I - \rho J_2^{-1})^{-1} D_2 \hat{z}_t. \tag{26}$$

To solve the second part of the system we use the upper portion of (17)

$$\left(\begin{bmatrix} x_{1t+1} \\ E_t x_{2t+1} \end{bmatrix} = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix} \begin{bmatrix} x_{1t} \\ x_{2t} \end{bmatrix} + \begin{bmatrix} E_1 \\ E_2 \end{bmatrix} f_t. \right)$$

$$x_{1t+1} = \tilde{A}_{11} x_{1t} + \tilde{A}_{12} x_{2t} + E_1 f_t, \tag{27}$$

where $\tilde{A} = \Lambda^{-1} J \Lambda$, and $\tilde{A}_{11}$, $\tilde{A}_{12}$ are conformable with x_{1t} and x_{2t}. Plugging the

[2] We use the Law of Iterated Expectations: $E_t[E_{t+1}(x_{t+2})] = E_t(x_{t+2})$ for any x_t.

expression for x_{2t} in equation (25) we obtain a solution for x_{1t}.

Note that the key requirement for this method to be applicable is the invertibility of the matrix A in

$$AE_t x_{t+1} + Bx_t + Cv_t = 0.$$

2 Sims's Method

An alternative (but similar) approach to solving the linear expectation systems was developed by Sims (2001). He proposes a solution method applied to models expressed as

$$Ax_t = Bx_{t-1} + Cv_t + D\eta_t + E, \tag{28}$$

where E is a vector of constants. Instead of expressing variables in terms of expected values, Sims drops expectations operators and introduces the expectations errors, η_{t+1}, defined as $x_{t+1} = E_t x_t + \eta_{t+1}$. Additionally, the exogenous shock is now incorporated into x_t (that is z_t is a part of x_t). [3]

Step 1:

Instead of Jordan decomposition, use 'QZ factorization' to decompose A and B as

$$A = Q'\Lambda Z' \tag{29}$$

$$B = Q'\Omega Z' \tag{30}$$

where (Q, Z) are unitary, and (Λ, Ω) are upper triangular.[4] Next, (Q, Z, Λ, Ω) are ordered such that, in absolute value, the generalized eigenvalues of A and B are organized in Λ and Ω in increasing order moving from left to right. These generalized eigenvalues, ϑ, of A and B are obtained as the solution to $Ae = \vartheta Be$, and ϑ can be calculated as the ratio of diagonal elements of A and B, $\vartheta_i = \frac{\lambda_{ii}}{\omega_{ii}}$. [5]

[3]Note that our model needs to be lagged by one period in order to match the notation (and code) of Sims.

[4]A unitary matrix Q satisfies $Q'Q = QQ' = I$. If Q and/or Z contain complex values, the transpositions reflect complex conjugation, that is, each complex entry is replaced by its conjugate and then transposed.

[5]Sims' website provides a program that computes and orders the eigenvalues appropriately.

Using these factorization, pre-multiply both sides of the equation by Q. The original system can be then expressed as

$$\Lambda z_t = \Omega z_{t-1} + QCv_t + QD\eta_t + QE, \tag{31}$$

where $z_{t+1} = Z' x_{t+1}$.[6]

Step 2:

Partition equation (31) into explosive and non-explosive blocks:

$$\begin{bmatrix} \Lambda_{11} & \Lambda_{12} \\ 0 & \Lambda_{22} \end{bmatrix} \begin{bmatrix} z_{1t} \\ z_{2t} \end{bmatrix} = \begin{bmatrix} \Omega_{11} & \Omega_{12} \\ 0 & \Omega_{22} \end{bmatrix} \begin{bmatrix} z_{1t-1} \\ z_{2t-1} \end{bmatrix} + \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} [Cv_t + D\eta_t + E]. \tag{32}$$

The model can now be solved iteratively.

Step 3:

To solve the lower (explosive) part of the system , z_{2t}, note that the lower block of (32) can be written as

$$\Lambda_{22} z_{2t} = \Omega_{22} z_{2t-1} + w_{2t}, \tag{33}$$

where $w_t = Q[Cv_t + D\eta_t + E]$ and w_{1t} and w_{2t} are partitioned conformably. Using equation (33) we can compute z_{2t} as

$$z_{2t} = M_{22} z_{2t+1} - \Omega_{22}^{-1} w_{2t+1}, \tag{34}$$

where $M = \Omega_{22}^{-1} \Lambda_{22}$. Iterating forward yields the expression for z_{2t}

$$z_{2t} = -\sum_{i=0}^{\infty} M^i \Omega_{22}^{-1} w_{2t+1+i}, \tag{35}$$

since $\lim_{t\to\infty} M^t z_{2t} = 0$. (Recall that $M = \Omega_{22}^{-1}\Lambda_{22}$ and generalized eigenvalues expressed as diagonal elements ω_{ii}/λ_{ii} are all outside unit circle in the lower block of the system.)

[6]Since Q is unitary $QQ' = I$.

Since w_t is defined as $w_t = Q[Cv_t + D\eta_t + E]$, equation (35) expresses z_{2t} as a function of future values of structural and expectational errors.

Since z_{2t} is known at time t, and $E_t(\eta_{t+s}) = E_t(v_{t+s}) = 0$ for $s > 0$ equation (35) can be written as

$$z_{2t} = -\sum_{i=0}^{\infty} M^i \Omega_{22}^{-1} Q_2 E_2, \tag{36}$$

where $Q_2 E_2$ are the lower portion of QE conformable with z_2.[7]

Since $-\sum_{i=0}^{\infty} M^i = -(I - M)^{-1}$, the solution of z_{2t} is obtained as

$$z_{2t} = (\Lambda_{22} - \Omega_{22})^{-1} Q_2 E_2. \tag{37}$$

Step 4:
The final step is to solve for z_{1t} in equation (32). Note that the solution of z_{1t} requires a solution for the expectations errors that appear in (32).

Sims notes that if there is a unique solution for the model, there exist a systematic relationship exists between the expectations errors associated with z_{1t} and z_{2t}.

Necessary and sufficient condition for uniqueness
If there exists a $k \times (n-k)$ matrix Φ that satisfies

$$Q_1 D = \Phi Q_2 D, \tag{38}$$

the equilibrium is unique.

Φ represents the systematic relationship between the expectations errors associated with z_{1t} and z_{2t} noted above.

If there exists a unique equilibrium we can calculate Φ, as in equation (38), and the solution of z_{1t} can be computed. If we pre-multiply (31)

$$\begin{bmatrix} \Lambda_{11} & \Lambda_{12} \\ 0 & \Lambda_{22} \end{bmatrix} \begin{bmatrix} z_{1t} \\ z_{2t} \end{bmatrix} = \begin{bmatrix} \Omega_{11} & \Omega_{12} \\ 0 & \Omega_{22} \end{bmatrix} \begin{bmatrix} z_{1t-1} \\ z_{2t-1} \end{bmatrix} + \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} [Cv_t + D\eta_t + E].$$

[7] Sims also considers the case in which the structural innovations v_t are serially correlated, which leads to a generalization of (36).

by $[I \;\; -\Phi]$, we obtain

$$\begin{bmatrix} \Lambda_{11} & \Lambda_{12}-\Phi\Lambda_{22} \end{bmatrix} \begin{bmatrix} z_{1t} \\ z_{2t} \end{bmatrix} = \begin{bmatrix} \Omega_{11} & \Omega_{12}-\Phi\Omega_{22} \end{bmatrix} \begin{bmatrix} z_{1t-1} \\ z_{2t-1} \end{bmatrix} + \begin{bmatrix} Q_1-\Phi Q_2 \end{bmatrix} [Cv_t + D\eta_t + E]. \tag{39}$$

If equation (38) holds, the loading factor for the expectational errors, η_t, is zero. Then the system may be written in the form

$$x_t = \Theta_e + \Theta_0 x_{t-1} + \Theta_1 v_t, \tag{40}$$

where

$$H = Z \begin{bmatrix} \Lambda_{11}^{-1} & -\Lambda_{11}^{-1}(\Lambda_{12}-\Phi\Lambda_{22}) \\ 0 & I \end{bmatrix} \tag{41}$$

$$\Theta_e = H \begin{bmatrix} Q_1-\Phi Q_2 \\ (\Omega_{22}-\Lambda_{22})^{-1}Q_2 \end{bmatrix} E \tag{42}$$

$$\Theta_0 = Z\Lambda_{11}^{-1}[\Omega_{11}(\Omega_{12}-\Phi\Omega_{22})]Z' \tag{43}$$

$$\Theta_1 = H \begin{bmatrix} Q_1-\Phi Q_2 \\ 0 \end{bmatrix} D \tag{44}$$