Lecture 2: The basic RBC model

Advanced Macroeconomics
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Plan of the Presentation

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RBC as benchmark

- The RBC model should be seen as a benchmark.
    - If a model with optimizing agents and instantaneous market clearing can explain the business cycle, no need for imperfections such as sticky prices to explain macroeconomic fluctuations.
- Focus on technology shocks allows to discern other shocks.
    - Growth theory: increases in TFP are the ultimate source of long-run growth in output per hour.

What will we do?

- Write down the problem
- Derive first order conditions
- Derive steady state
- Log-linearize the equations
- Solve
- Check stability
- Simulate

Plan of the Presentation

Model

- DSGE models (so also RBC) are derived from microeconomic problems
- Write down the problems
  1. households maximize utility subject to budget constraint,
  2. producers maximize profits subject to technology,
  3. markets clear.
- Solve it, e.g. construct Lagrangeans (or Bellman equations).

Today's solution

- Derive first order conditions for optimum
- Get rid of Lagrange multipliers
- *(Log-)*linearize all equations (optional)
- Solve the system
- Set parameters
- Check stability

Households - the problem

- A representative household maximizes lifetime utility

$$E_t \sum_{i=0}^{\infty} \beta^{t+i} u(c_{t+i}, l_{t+i})$$

subject to the budget constraint

$$c_t + i_t = w_t l_t + r_t k_{t-1} + Div_t$$

and the capital accumulation rule

$$k_t = (1-\delta) k_{t-1} + i_t.$$

- The household rents labor l_t, and capital, k_t, to firms and receives as compensation the real wage w_t, the rental rate r_t; and dividends, Div_t.

Households - Preferences

Choosing preferences, $u(c, l)$:

- concave, strictly increasing, twice continuously differentiable ($\mathcal{C}^2$) in both arguments,
- satisfy Inada conditions, $\lim_{c\to 0} u_c(c,l) = \lim_{n\to 0} u_l(c, 1-n) = \infty$,
- consistent with economic theory and empirical observation:
  1. relatively constant risk premium,
  2. relatively constant consumption growth,
  3. stationary hours worked.
- King, Plosser, and Rebelo (1988a,b): utility must be of form

$$u(c,n) = \begin{cases} \frac{(cv(l)^{1-\sigma})-1}{1-\sigma}, & \text{for} \quad \sigma > 0, \sigma \neq 1 \\ \ln c + \ln v(l), & \text{for} \quad \sigma = 1. \end{cases}$$

with some restrictions imposed on $v(l)$:

- $v(l)$ is $\mathcal{C}^2$
- monotonicity and concavity/convexity depends on σ

Households - Preferences

Choosing preferences, $u(c,l)$:

- for now we assume $\psi = 1$ and

$$u(c_t, l_t) = \frac{c_t^{1-\sigma}}{1-\sigma} - \frac{l_t^{1+\varphi}}{1+\varphi}$$

and, if $\sigma \to 1$,

$$u(c_t, l_t) = \ln c_t - \frac{l_t^{1+\varphi}}{1+\varphi}.$$

- (The log-log formulation could be given by

$$u(c_t, l_t) = \ln c_t + \psi ln(1 - l_t)$$

)

Lagrangean

- We can substitute for investment

$$c_t + k_t = w_t l_t + (1 + r_t - \delta)k_{t-1} + Div_t$$

where $r_t - \delta$ is the real interest rate.

- Then, the household chooses c_t, l_t and k_{t+1} to maximize lifetime utility subject to budget constraints.
- Write down the Lagrangean:

$$\mathcal{L}_t = E_t \sum_{i=0}^{\infty} \Bigg[\beta^{t+i} \left(\frac{c_{t+i}^{1-\sigma}}{1-\sigma} - \frac{l_{t+i}^{1+\varphi}}{1+\varphi} \right)$$
$$- \lambda_{t+i} \big(c_{t+i} + k_{t+i} - w_{t+i} l_{t+i} - (1 + r_{t+i} - \delta) k_{t-1+i} + Div_t \big) \Bigg]$$

where λ_{t+i} denotes a Lagrange multiplier associated with budget constraint in period $t+i$.

First order conditions

$$\mathcal{L}_t = E_t \sum_{i=0}^{\infty} \Bigg[\beta^{t+i} \left(\frac{c_{t+i}^{1-\sigma}}{1-\sigma} - \frac{l_{t+i}^{1+\varphi}}{1+\varphi} \right)$$

$$-\lambda_{t+i} \left(c_{t+i} + k_{t+i} - w_{t+i} l_{t+i} - (1 + r_{t+i} - \delta) k_{t-1+i} + Div_t \right) \Bigg]$$

$$c_t : \frac{\partial \mathcal{L}}{\partial c_t} = \beta^t c_t^{-\sigma} - \lambda_t = 0$$

$$l_t : \frac{\partial \mathcal{L}}{\partial l_t} = -\beta^t l_t^{\varphi} + \lambda_t w_t = 0$$

$$k_t : \frac{\partial \mathcal{L}}{\partial k_t} = -\lambda_t + E_t \left[\lambda_{t+1} (1 + r_{t+i} - \delta) \right] = 0$$

- Note that additionally we have a transversality condition (TVC), $lim_{t \to \infty} \lambda_t k_t = 0$,
  i.e. the value (discounted into present-value utils) of each additional unit of capital at infinity times the actual amount of capital has to be zero.

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$$\left. - \lambda_{t+i} \left(c_{t+i} + k_{t+i} - w_{t+i} l_{t+i} - (1 + r_{t+i} - \delta) k_{t-1+i} + Div_t \right) \right]$$

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Equilibrium conditions - Euler equation

Use $\partial\mathcal{L}/\partial c_t = 0$ to substitute for λ in $\partial\mathcal{L}/\partial k_t = 0$ to get

$$c_t^{-\sigma} = \beta E_t[c_{t+1}^{-\sigma}(1 + r_{t+1} - \delta)]$$

- This is the **consumption Euler equation**.
- It determines the household's *intertemporal choic*e (how much to consume today, how much to save).
- In equilibrium the disutility from one unit less consumed today equals expected discounted utility of consuming $(1 + r_{t+1} - \delta)$ units tomorrow.

Equilibrium conditions - consumption vs. leisure

Use $\partial\mathcal{L}/\partial c_t = 0$ to substitute for λ in $\partial\mathcal{L}/\partial l_t = 0$ to get

$$\frac{l_t^{\varphi}}{c_t^{-\sigma}} = w_t$$

- This equation determines the household's *intratemporal choice*: how much to consume *against* how much to work.
- In equilibrium the utility of one unit more of work should be equal to the utility from consuming the compensation (real wage).

Firms - the problem

- A representative firm uses capital and labor hired from households to produce a unique good y_t.

- Its objective is to maximize profits:

$$Div_t = y_t - w_t l_t - r_t k_{t-1}$$

- subject to technology

$$y_t = z_t k_{t-1}^{\alpha} l_t^{1-\alpha},$$

where z_t denotes aggregate productivity.

Firms - equilibrium condition for labor

- Substitute for y_t to get:

$$Div_t = z_t k_{t-1}^{\alpha} l_t^{1-\alpha} - w_t l_t - r_t k_{t-1}$$

- First order condition for labor is:
l_t :

$$\frac{\delta Div_t}{\delta l_t} = (1-\alpha) z_t k_{t-1}^{\alpha} l_t^{-\alpha} - w_t = 0$$

- In equilibrium the marginal product of labor equals the real wage
- To simplify use production function to write:

$$(1-\alpha)\frac{y_t}{l_t} = w_t$$

Firms - equilibrium condition for labor

- Substitute for y_t to get:

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- To simplify use production function to write:

$$(1-\alpha)\frac{y_t}{l_t} = w_t$$

Firms - equilibrium condition for capital

- First order condition for capital is:
  k_{t-1} :

$$\frac{\delta Div_t}{\delta l_t} = \alpha z_t k_{t-1}^{\alpha-1} l_t^{1-\alpha} - r_t = 0$$

- In equilibrium the marginal product of capital equals the real rental rate of capital.

- Using production function we write:

$$\alpha \frac{y_t}{k_{t-1}} = r_t$$

- In the RBC setting profits are zero (perfect competition), $Div_t = 0$.
  To see this substitute r_t and w_t into the profit function.

Firms - equilibrium condition for capital

- First order condition for capital is:
  k_{t-1} :
  $$\frac{\delta Div_t}{\delta l_t} = \alpha z_t k_{t-1}^{\alpha-1} l_t^{1-\alpha} - r_t = 0$$

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- Using production function we write:
  $$\alpha \frac{y_t}{k_{t-1}} = r_t$$

- In the RBC setting profits are zero (perfect competition), $Div_t = 0$.
  To see this substitute r_t and w_t into the profit function.

Productivity and market clearing

- It is assumed that productivity z_t follows an $AR(1)$ process:

$$z_t = exp(\epsilon_t) z_{t-1}^{\rho},$$

or

$$\hat{z}_t = \rho \hat{z}_{t-1} + \epsilon_t,$$

where ϵ_t is a productivity shock

- This is the only stochastic process in the basic RBC model
- Together with the internal persistence of the model it generates the business cycle
- The goods market clears:

$$c_t + i_t = y_t$$

Equilibrium conditions - summary

- Now we have a system of 8 equations with 8 endogenous variables (c, r, l, w, k, i, y, z):

$$c_t^{-\sigma} = \beta E_t[c_{t+1}^{-\sigma}(1 + r_{t+1} - \delta)] \tag{1}$$

$$w_t = \frac{l_t^{\varphi}}{c_t^{-\sigma}} \tag{2}$$

$$k_t = (1 - \delta)k_{t-1} + i_t \tag{3}$$

$$y_t = c_t + i_t \tag{4}$$

Equilibrium conditions - cont'd

$$y_t = z_t k_{t-1}^{\alpha} l_t^{1-\alpha} \quad (5)$$

$$w_t = (1-\alpha)\frac{y_t}{l_t} \quad (6)$$

$$r_t = \alpha \frac{y_t}{k_{t-1}} \quad (7)$$

$$z_t = exp(\epsilon_t) z_{t-1}^{\rho} \quad (8)$$

- This can be solved
- But two problems arise
  1. equations are non-linear
  2. these are expectational difference equations

Plan of the Presentation

Interest rate and productivity

- In the deterministic steady state uncertainty disappears and all variables are constant, for example $c_t = c_{t+1} = c^{ss}$.
- We assume that in the steady state z is constant and

$$z^{ss} = 1 \tag{9}$$

- From (1) we get

$$c_t^{-\sigma} = \beta E_t[c_{t+1}^{-\sigma}(1 + r_{t+1} - \delta)]$$

$$(c^{ss})^{-\sigma} = \beta (c^{ss})^{-\sigma}(1 + r^{ss} - \delta)$$

Simplifying

$$r^{ss} = \beta^{-1} - (1 - \delta) \tag{10}$$

Capital-labor and investment-capital ratios

- Substituting form the production function (5) for y_t into (7) we get

$$\alpha \frac{z_t k_{t-1}^{\alpha} l_t^{1-\alpha}}{k_{t-1}} = r_t$$

- In the steady state

$$r^{ss} = \alpha (k^{ss})^{\alpha-1} (l^{ss})^{1-\alpha} = \alpha \left(\frac{k^{ss}}{l^{ss}} \right)^{\alpha-1}$$

Rearranging we can obtain formula for capital-labor ratio

$$\frac{k^{ss}}{l^{ss}} = \left(\frac{r^{ss}}{\alpha} \right)^{\frac{1}{\alpha-1}} \tag{11}$$

where r^{ss} is given by (10).

Wage and investment-capital ratio

- Substituting form the production function (5) for y_t into (6) we get

$$w_t = (1-\alpha)\frac{z_t k_{t-1}^{\alpha} l_t^{1-\alpha}}{l_t} = (1-\alpha) z_t \frac{k_{t-1}^{\alpha}}{l_t^{\alpha}}$$

which in the steady state becomes

$$w^{ss} = (1-\alpha)\left(\frac{k^{ss}}{l^{ss}}\right)^{\alpha} \tag{12}$$

where k^{ss}/l^{ss} is given by (11).

- From (3) in the steady state we have

$$k^{ss} = (1-\delta)k^{ss} + i^{ss}$$

which simplifies to

$$\frac{i^{ss}}{k^{ss}} = \delta \tag{13}$$

Consumption-labor ratio

- Substituting from (5) for y_t into (4) we get

$$c_t + i_t = z_t k_{t-1}^{\alpha} l_t^{1-\alpha}$$

which in the steady state becomes

$$c^{ss} + i^{ss} = (k^{ss})^{\alpha} (l^{ss})^{1-\alpha} = \left(\frac{k^{ss}}{l^{ss}}\right)^{\alpha} l^{ss}$$

- Dividing by l^{ss}

$$\frac{c^{ss}}{l^{ss}} + \frac{i^{ss}}{l^{ss}} = \left(\frac{k^{ss}}{l^{ss}}\right)^{\alpha}$$

- Substituting for i^{ss} from (13) and rearranging

$$\frac{c^{ss}}{l^{ss}} = \left(\frac{k^{ss}}{l^{ss}}\right)^{\alpha} - \delta \frac{k^{ss}}{l^{ss}} \tag{14}$$

where k^{ss}/l^{ss} is given by (11).

Labor

- From (2) we have

$$\frac{l_t^{\varphi}}{c_t^{-\sigma}} = w_t$$

which in the steady state it becomes

$$(l^{ss})^{\varphi}(c^{ss})^{\sigma} = w^{ss}$$

Substituting for w^{ss} from (12) and for c^{ss} from (14)

$$(l^{ss})^{\varphi}\left[\left(\frac{k^{ss}}{l^{ss}}\right)^{\alpha} - \delta\frac{k^{ss}}{l^{ss}}\right]^{\sigma}(l^{ss})^{\sigma} = (1-\alpha)\left(\frac{k^{ss}}{l^{ss}}\right)^{\alpha}$$

Solving with respect to l^{ss} we get

$$l^{ss} = \left[\frac{(1-\alpha)\left(\frac{k^{ss}}{l^{ss}}\right)^{\alpha}}{\left[\left(\frac{k^{ss}}{l^{ss}}\right)^{\alpha} - \delta\frac{k^{ss}}{l^{ss}}\right]^{\sigma}}\right]^{\frac{1}{\varphi+\sigma}} \tag{15}$$

where k^{ss}/l^{ss} is given by (11).

Remaining variables

- To obtain c^{ss}, note
$$c^{ss} = \frac{c^{ss}}{l^{ss}} l^{ss} \tag{16}$$
where c^{ss}/l^{ss} is given by (14) and l^{ss} is given by (15).
- We can obtain capital k^{ss} from
$$k^{ss} = \frac{k^{ss}}{l^{ss}} l^{ss} \tag{17}$$
where k^{ss}/l^{ss} is given by (11) and l^{ss} is given by (15).
- To obtain output we use the production function
$$y^{ss} = (k^{ss})^{\alpha} (l^{ss})^{1-\alpha}$$
where k^{ss} is given by (17) and l^{ss} is given by (15).
- and we can obtain investment form (13)
$$i^{ss} = \delta k^{ss}$$
where k^{ss} is given by (17).

Plan of the Presentation

Non-linear vs. linear models

- We have a system of non-linear difference equations

- Two options:
  1. use Dynare to linearize it,
  2. linearize by hand.

- Linearizing by hand can be tedious but has some advantages:
  1. some parameters may disappear
  2. the system is easier to understand
  3. solution in Dynare easier (steady state is known)