Advanced Macroeconomics: Final Exam

WNE UW - Spring 2017

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Answer all questions. The due date is the July 1, 2017, 15:59. Email me if something is not clear. All answers must be type in. Good luck !

Question 1 Stochastic Solow Model
Let time be discrete, $t = 0, 1, \ldots$ Let the national resource constraint be

$$c_t + i_t = y_t = z_t f(k_t)$$

where c_t denotes consumption, it denotes investment, y_t denotes output, and k_t denotes capital. The production function has the following properties:

$$\begin{aligned} f(0) &= 0, \\ f'(k) &> 0\,, \qquad f''(k) < 0, \\ \lim_{k \to 0} f'(k) &= \infty\,, \quad \lim_{k \to \infty} f'(k) = 0\,. \end{aligned}$$

The production function features random *iid* technology shocks $\{z_t\}$ with properties to be described below and with a given initial condition z_0.

Let capital accumulation be given by

$$k_{t+1} = i_t, \qquad k_0 \text{ given,}$$

i.e., there is "full depreciation". Finally, let consumption be a fixed fraction of national output

$$c_t = (1 - s) y_t, \qquad 0 < s < 1\,,$$

where s denotes the national saving rate.

1. Show that this model can be reduced to a single non-linear stochastic difference equation in k_t and z_t, i.e., that you can write the model as

   $$k_{t+1} = \psi(k_t, z_t), \qquad k_0 \text{ and } z_0 \text{ given.}$$

   Provide an explicit formula for the function ψ.

2. Let $z_t = \bar{z} = 1$ always and let $\bar{k}$ denote a solution to

$$\bar{k} = \psi(\bar{k}, 1).$$

How many such $\bar{k}$ are there? Linearize the function $\psi(k_t, 1)$ around each of these points and determine the local stability or instability of each such point.

3. Suppose that the production function is Cobb-Douglas with capital share α,

$$f(k) = k^{\alpha}, \qquad 0 < \alpha < 1 \; .$$

Provide an explicit solution for each $\bar{k}$ (continuing to hold $z_t = \bar{z} = 1$). Explain how the fixed points depend on the parameters α and s. Give economic interpretations.

4. Suppose that $\log(z_t)$ are *iid* Gaussian with mean 0 and variance σ^2. Let

$$\hat{x}_t = \log\left(\frac{x_t}{\bar{x}}\right)$$

be the log deviation of some variable from its "non-stochastic steady state". Log-linearize $\psi(k_t, z_t)$ to derive an approximate linear stochastic difference equation in the state $\hat{k}_t$ and the shocks $\hat{z}_t$ (assume as above that $f(k)$ is Cobb-Douglas). Solve for the stationary distribution of $\hat{k}_t$ and explain how its mean and variance depend on the parameters α, s, and σ. Give economic interpretations.
(Hint: use the fact that the stationary distribution of $\hat{x}_t$ implies that unconditional mean and variance are constant, i.e. $E(\hat{x}_{t+i}) = E(\hat{x}_t)$ and $Var(\hat{x}_{t+i}) = Var(\hat{x}_t)$ for all i.)

Question 2 (Value Function)
Consider a consumer with utility function

$$\sum_{t=0}^{\infty} \beta^t U(c_t), \qquad 0 < \beta < 1.$$

The consumer is endowed with a cake of size x_0 at time $t = 0$. Each period, she has cake x_t and can either consume some, c_t, or hold some cake over to next period, x_{t+1}.

1. Provide a dynamic programming representation of this problem. In your answer, let $V(x)$ denote the utility value of a cake of size x.

2. Let the period utility function be

$$U(c) = \frac{c^{1-\sigma}}{1-\sigma}, \qquad \sigma > 0.$$

Guess that the value function $V(x)$ and policy function $g(x)$ for your dynamic programming problem have the forms

$$V(x) = \alpha \frac{x^{1-\sigma}}{1-\sigma}$$
$$g(x) = \theta x$$

for some unknown coefficients $\alpha > 0$ and $0 < \theta < 1$. Solve for the unknown coefficients.

3. Verify that over the infinite time horizon, the consumer eats all the cake, namely

$$\sum_{t=0}^{\infty} c_t = x_0$$

At what rate does the cake diminish? Provide a formula that calculates how long it takes for there to only be $\epsilon > 0$ crumbs of cake left. How does this rate of consumption depend on the parameters β and σ? Give economic intuition.

Question 3 (OLG)

Consider a three period OLG endowment economy with logarithmic preferences

$$U = \ln c_t(t) + \beta \ln c_t(t+1) + \beta^2 \ln c_t(t+2),$$

where $c_t(t+i)$ with $i = 0, 1, 2$ denotes the consumption at time $t+i$ of an agent born at period t and $\beta \in (0, 1]$ is the discount factor. Denote by $e = (e_0, e_1, e_2)$ lifetime endowment/income profile of agents, i.e. $e_i > 0$ denotes endowments of agent in stage i of their life.

The budget constraints are

$$a_t(t) + c_t(t) \leq e_0$$
$$a_t(t+1) + c_t(t+1) \leq e_1 + R(t)a_t(t)$$
$$c_t(t+2) \leq e_2 + R(t+1)a_t(t+1)$$

where $a_t(t+i)$ denotes the real value of assets saved/stored at period $t+i$ of an agent born at period t and $R(t)$ denotes the interest rate/rate of return on asset saved/stored on period t.

Fiat money is the only asset. The money stock, $H(t)$, grows at a constant rate θ

$$H(t) = \theta H(t-1).$$

1. Eliminate $a_t(t+i)$ and write down the life-time/consolitdate budget constraint.

2. Set-up the Lagrangian and write the first-order conditions.

3. Write down the solution for $c_t(t+i),\ i = 0, 1, 2$ in terms of endowments e_0, e_1, e_2 and interest rates $R(t), R(t+1)$.

4. Write down individual $(a_t(t+i))$ and aggregate $(A(t))$ asset holdings.
(Hint: $A(t) = a_t(t) + a_{t-1}(t)$.)

5. Using asset market clearing condition:

$$A(t) = \frac{H(t)}{P(t)},$$

where $R(t) = \frac{P(t)}{P(t+1)}$ and $P(t)$ denotes price level in period t, write down the equilibrium condition in that economy.
(Hint: you can eliminate $H(t)$ to obtain a difference equation in $R(t)$.)

6. Consider a steady state $R(t) = R$ for all t. What is the condition for a valued fiat currency steady-state (i.e. with $A(t) = A > 0$) to exists?

7. How does this condition look like for constant money supply (i.e. $\theta = 1$) and $\beta = 1$?