

Lecture: Solving Linear DSGE Models

We have a set of linear expectational equations:

$$AE_t x_{t+1} + Bx_t + Cv_{t+1} = 0. \quad (\text{A})$$

We seek a solution of the form

$$x_{t+1} = Fx_t + Gv_{t+1}. \quad (\text{B})$$

This solution represents the time series behavior of $\{x_t\}$ as a function of $\{v_t\}$, where v_t is a vector of exogenous innovations, or as frequently referenced, structural shocks.

In this note we use a standard timing for capital, that is

$$y_t = z_t k_t^\alpha l_t^{1-\alpha}$$

for production function,

$$k_{t+1} = (1 - \delta)k_t + i_t$$

for capital accumulation equation,

$$Div_t = y_t - w_t l_t - r_t k_t$$

for profit/dividend function, and

$$c_t + i_t = w_t l_t + r_t k_t + Div_t$$

for the budget constraint.

The linearized equation with these equations are then as follows.

Equation 1:

$$\sigma(E_t \hat{c}_{t+1} - \hat{c}_t) = (1 - \beta(1 - \delta))E_t \hat{r}_{t+1} \quad (1)$$

Equation 2:

$$\varphi \hat{l}_t + \sigma \hat{c}_t = \hat{w}_t \quad (2)$$

Equation 3:

$$\hat{k}_{t+1} = (1 - \delta)k_t + \frac{i^{ss}}{k^{ss}} \hat{i}_t \quad (3)$$

Equation 4:

$$\left(1 - \frac{i^{ss}}{y^{ss}}\right) \hat{c}_t + \frac{i^{ss}}{y^{ss}} \hat{i}_t = \hat{y}_t \quad (4)$$

Equation 5:

$$\hat{y}_t = \hat{z}_t + \alpha \hat{k}_t + (1 - \alpha) \hat{l}_t \quad (5)$$

Equation 6:

$$\hat{y}_t - \hat{l}_t = \hat{w}_t \quad (6)$$

Equation 7:

$$\hat{y}_t - \hat{k}_t = \hat{r}_t \quad (7)$$

Equation 8:

$$\hat{z}_t = \rho \hat{z}_{t-1} + \epsilon_t \quad (8)$$

Our original system has 8 equations but to we can reduce the system by eliminating some variables/equations.

First, equations (2) and (6) gives

$$\hat{l}_t = \frac{1}{1 + \varphi} \hat{y}_t - \frac{\sigma}{1 + \varphi} \hat{c}_t$$

Putting equation (5) and expression for \hat{l}_t yields

$$\begin{aligned} \hat{y}_t &= z_t + \alpha \hat{k}_t + (1 - \alpha) \left(\frac{1}{1 + \varphi} \hat{y}_t - \frac{\sigma}{1 + \varphi} \hat{c}_t \right) \\ \left(1 - \frac{1 - \alpha}{1 + \varphi}\right) \hat{y}_t &= z_t + \alpha \hat{k}_t - \frac{(1 - \alpha)\sigma}{1 + \varphi} \hat{c}_t \\ \hat{y}_t &= \frac{1 + \varphi}{\alpha + \varphi} z_t + \frac{1 + \varphi}{\alpha + \varphi} \alpha \hat{k}_t - \frac{(1 - \alpha)\sigma}{\alpha + \varphi} \hat{c}_t \end{aligned}$$

Taking from equation (4) formula for \hat{i}_t and putting it into equation (3) yields

$$\hat{k}_{t+1} = (1 - \delta)k_t + \frac{y^{ss}}{k^{ss}}\hat{y}_t - \frac{c^{ss}}{k^{ss}}\hat{c}_t$$

and using expression for \hat{y}_t ,

$$\begin{aligned}\hat{k}_{t+1} - (1 - \delta)k_t &= -\frac{c^{ss}}{k^{ss}}\hat{c}_t + \frac{y^{ss}}{k^{ss}}\hat{y}_t \\ \hat{k}_{t+1} - (1 - \delta)k_t &= -\frac{c^{ss}}{k^{ss}}\hat{c}_t + \frac{y^{ss}}{k^{ss}}\left(\frac{1 + \varphi}{\alpha + \varphi}\hat{z}_t + \frac{1 + \varphi}{\alpha + \varphi}\alpha\hat{k}_t - \frac{(1 - \alpha)\sigma}{\alpha + \varphi}\hat{c}_t\right)\end{aligned}$$

Equation (1) can be written as

$$\begin{aligned}\sigma(E_t\hat{c}_{t+1} - \hat{c}_t) &= \beta r^{ss} E_t \hat{r}_{t+1} \\ \sigma(E_t\hat{c}_{t+1} - \hat{c}_t) &= \beta r^{ss} E_t \hat{y}_{t+1} - \beta r^{ss} E_t \hat{k}_{t+1}\end{aligned}$$

which, using expression for y_t above, can be written as

$$\sigma(E_t\hat{c}_{t+1} - \hat{c}_t) = -\beta r^{ss} E_t \hat{k}_{t+1} + \beta r^{ss} \frac{1 + \varphi}{\alpha + \varphi} E_t \hat{z}_{t+1} + \beta r^{ss} \frac{1 + \varphi}{\alpha + \varphi} \alpha E_t \hat{k}_{t+1} - \beta r^{ss} \frac{(1 - \alpha)\sigma}{\alpha + \varphi} E_t \hat{c}_{t+1}$$

or

$$-\sigma\hat{c}_t + \sigma\left(1 + \beta r^{ss} \frac{(1 - \alpha)}{\alpha + \varphi}\right) E_t \hat{c}_{t+1} = \beta r^{ss} \left(\frac{1 + \varphi}{\alpha + \varphi} \alpha - 1\right) E_t \hat{k}_{t+1} + \beta r^{ss} \frac{1 + \varphi}{\alpha + \varphi} E_t \hat{z}_{t+1}.$$

Note that together with equation (8)

$$\hat{z}_t = \rho\hat{z}_{t-1} + \epsilon_t$$

we have transform our original set of equations into the system of 3 equations with 3 endogenous variables (c, k, z) , and 1 exogenous variable (ϵ) .

We can write it as

$$\kappa_{11}\hat{c}_t + \kappa_{12}E_t\hat{k}_{t+1} + \kappa_{13}E_t\hat{z}_{t+1} + \kappa_{14}E_t\hat{c}_{t+1} = 0 \quad (9)$$

$$\kappa_{21}\hat{c}_t + \kappa_{22}\hat{k}_t + \kappa_{23}\hat{z}_t + \kappa_{24}\hat{k}_{t+1} = 0 \quad (10)$$

$$\hat{z}_{t+1} + \rho\hat{z}_t = \epsilon_{t+1} \quad (11)$$

where κ_{ij} contain the ‘deep’ parameters of the model.

Setting $x_t = \{k_t, c_t\}$ and $v_t = z_t$ the model has the form consistent with equation (A):

$$AE_t x_{t+1} + Bx_t + Cv_{t+1} = 0.$$

with A , B , and C being functions of κ s.

(Alternatively, we could set $x_t = \{z_t, k_t, c_t\}$ and $v_t = \epsilon_t$ and solve the model.)

1 Blanchard and Kahn’s Method

Blanchard and Kahn(1980) developed the method to solve models written as

$$\begin{bmatrix} x_{1t+1} \\ E_t x_{2t+1} \end{bmatrix} = \tilde{A} \begin{bmatrix} x_{1t} \\ x_{2t} \end{bmatrix} + E f_t, \quad (12)$$

where the model’s variables have been divided into an $n \times 1$ vector of endogenous pre-determined variables x_{1t} (defined as variables for which $E_t x_{1t+1} = x_{1t+1}$), and an $m \times 1$ vector of endogenous non-predetermined variables x_{2t} . The $k \times 1$ vector f_t contains exogenous forcing variables.

For our model to conform to this specification we need to pre-multiply the entire system by A^{-1} so that

$$E_t x_{t+1} = -A^{-1}Bx_t - A^{-1}Cv_{t+1}.$$

which is possible only if A is invertible. Then $\tilde{A} = -A^{-1}B$ and $E = -A^{-1}C$.

In $x_t = \{\hat{c}_t, \hat{k}_t\}$, \hat{k}_t is predetermined (given \hat{k}_t and $\hat{i}_t = \hat{y}_t - \hat{c}_t$, \hat{k}_{t+1} is determined

by (10)); \hat{c}_t is endogenous but not predetermined (as indicated in (9)); and \hat{z}_t is an exogenous forcing variable. Thus in the notation of (12), we seek a specification of the model in the form

$$\begin{bmatrix} \hat{k}_{t+1} \\ E_t \hat{c}_{t+1} \end{bmatrix} = \tilde{A} \begin{bmatrix} \hat{k}_t \\ \hat{c}_t \end{bmatrix} + E \hat{z}_t. \quad (13)$$

To solve this system, Blanchard and Khan start with a Jordan decomposition to obtain Jordan normal form of \tilde{A} :

$$\tilde{A} = \Lambda^{-1} J \Lambda. \quad (14)$$

where J is a diagonal matrix with the eigenvalues of \tilde{A} along its leading diagonal and zeros in the off-diagonal elements, and Λ is a matrix of the corresponding eigenvectors. Eigenvalues are ordered in increasing absolute value in moving from left to right.¹ In particular, J can be written as

$$J = \begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix}, \quad (15)$$

where the eigenvalues in J_1 lie on or within the unit circle (i.e. of modulus less than one), and those in J_2 lie outside of the unit circle. J_2 is said to be unstable or explosive, since J_2^n diverges as n increases.

Before we proceed, we partition matrices Λ and E

$$\Lambda = \begin{bmatrix} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{21} & \Lambda_{22} \end{bmatrix}, \quad E = \begin{bmatrix} E_1 \\ E_2 \end{bmatrix}, \quad (16)$$

where Λ_{11} is conformable with J_1 , etc.

Blanchard-Khan condition

The Blanchard-Khan condition states that if the number of explosive eigenvalues is equal to the number of non-predetermined variables, the system is said to be saddle-

¹Eigenvalues of a matrix M are obtained from the solution of equations of the form $Me = \lambda e$, where e is an eigenvector and λ the associated eigenvalue.

path stable and a unique solution to the model exists.

On the other hand, if the number of explosive eigenvalues (i.e. greater than 1) exceeds the number of non-predetermined variables then no solution exists (and the system is said to be a source). Finally, if the number of explosive eigenvalues is less than the number of non-predetermined variables, then an infinity of solutions exist (and the system is said to be a sink).

If Blanchard-Khan condition holds then under the Jordan decomposition of \tilde{A} , (12) yields

$$\begin{bmatrix} x_{1t+1} \\ E_t x_{2t+1} \end{bmatrix} = \Lambda^{-1} J \Lambda \begin{bmatrix} x_{1t} \\ x_{2t} \end{bmatrix} + \begin{bmatrix} E_1 \\ E_2 \end{bmatrix} f_t. \quad (17)$$

Pre-multiplying by Λ we obtain

$$\begin{bmatrix} \bar{x}_{1t+1} \\ E_t \bar{x}_{2t+1} \end{bmatrix} = \begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix} \begin{bmatrix} \bar{x}_{1t} \\ \bar{x}_{2t} \end{bmatrix} + \begin{bmatrix} D_1 \\ D_2 \end{bmatrix} f_t. \quad (18)$$

where $\bar{x}_t = \Lambda x_t$ and $D = \Lambda E$, that is

$$\begin{bmatrix} \bar{x}_{1t} \\ \bar{x}_{2t} \end{bmatrix} = \begin{bmatrix} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{21} & \Lambda_{22} \end{bmatrix} \begin{bmatrix} x_{1t} \\ x_{2t} \end{bmatrix} \quad (19)$$

$$\begin{bmatrix} D_1 \\ D_2 \end{bmatrix} = \begin{bmatrix} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{21} & \Lambda_{22} \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \end{bmatrix}. \quad (20)$$

Under this transformation the two sets of equations are now decoupled, so that the non-predetermined variables depend only on the unstable eigenvalues of \tilde{A} contained in J_2 , as expressed in the lower part of (18). In other words, we can write each time $t + 1$ variable as solely a function of predetermined variables, exogenous variables and controls at time t .

To find the solution for the non-predetermined variables, we iterate forward lower portion of (18) as follows. Since

$$E_t \bar{x}_{2t+1} = J_2 \bar{x}_{2t} + D_2 f_{2t}$$

we can write \bar{x}_{2t} as

$$\bar{x}_{2t} = J_2^{-1} E_t \bar{x}_{2t+1} - J_2^{-1} D_2 f_{2t}. \quad (21)$$

Iterating it forward

$$\bar{x}_{2t+1} = J_2^{-1} E_{t+1} \bar{x}_{2t+2} - J_2^{-1} D_2 f_{2t+1}, \quad (22)$$

and substituting into (21) yields²

$$\bar{x}_{2t} = J_2^{-2} E_t \bar{x}_{2t+2} - J_2^{-2} D_2 E_t f_{2t+1} - J_2^{-1} D_2 f_{2t}. \quad (23)$$

Since J_2 contains eigenvalues that are outside the unit root, $\lim_{n \rightarrow \infty} J_2^{-n} = 0$. Therefore, iterating the substitution gives the following expression

$$\bar{x}_{2t} = - \sum_{i=0}^{\infty} J_2^{-(i+1)} D_2 E_t f_{2t+i}. \quad (24)$$

Using equation (19),

$$(\bar{x}_{2t} = \Lambda_{21} x_{1t} + \Lambda_{22} x_{2t},)$$

we can map it back to x_{2t} as

$$x_{2t} = -\Lambda_{22}^{-1} \Lambda_{21} x_{1t} - \Lambda_{22} \sum_{i=0}^{\infty} J_2^{-(i+1)} D_2 E_t f_{2t+i}. \quad (25)$$

In the case of our model, $E_t(f_{2t+i}) = \rho^i \hat{z}_t$, and thus (25) becomes

$$x_{2t} = -\Lambda_{22}^{-1} \Lambda_{21} x_{1t} - \Lambda_{22} J_2^{-1} (I - \rho J_2^{-1})^{-1} D_2 \hat{z}_t. \quad (26)$$

To solve the second part of the system we use the upper portion of (17)

$$\left(\begin{bmatrix} x_{1t+1} \\ E_t x_{2t+1} \end{bmatrix} = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix} \begin{bmatrix} x_{1t} \\ x_{2t} \end{bmatrix} + \begin{bmatrix} E_1 \\ E_2 \end{bmatrix} f_t \right)$$

$$x_{1t+1} = \tilde{A}_{11} x_{1t} + \tilde{A}_{12} x_{2t} + E_1 f_t, \quad (27)$$

where $\tilde{A} = \Lambda^{-1} J \Lambda$, and \tilde{A}_{11} , \tilde{A}_{12} are conformable with x_{1t} and x_{2t} . Plugging the

²We use the Law of Iterated Expectations: $E_t[E_{t+1}(x_{t+2})] = E_t(x_{t+2})$ for any x_t .

expression for x_{2t} in equation (25) we obtain a solution for x_{1t} .

Note that the key requirement for this method to be applicable is the invertibility of the matrix A in

$$AE_t x_{t+1} + Bx_t + Cv_t = 0.$$

2 Sims's Method

An alternative (but similar) approach to solving the linear expectation systems was developed by Sims (2001). He proposes a solution method applied to models expressed as

$$Ax_t = Bx_{t-1} + Cv_t + D\eta_t + E, \quad (28)$$

where E is a vector of constants. Instead of expressing variables in terms of expected values, Sims drops expectations operators and introduces the expectations errors, η_{t+1} , defined as $x_{t+1} = E_t x_t + \eta_{t+1}$. Additionally, the exogenous shock is now incorporated into x_t (that is z_t is a part of x_t).³

Sim's method involves following steps.

Step 1:

Instead of Jordan decomposition, use 'QZ factorization' (generalized Schur decomposition) to decompose A and B as

$$A = Q'\Lambda Z' \quad (29)$$

$$B = Q'\Omega Z' \quad (30)$$

where (Q, Z) are unitary, and (Λ, Ω) are upper triangular.⁴ Next, (Q, Z, Λ, Ω) are ordered such that, in absolute value, the generalized eigenvalues of A and B are organized in Λ and Ω in increasing order moving from left to right. These generalized eigenvalues, ϑ , of A and B are obtained as the solution to $Ae = \vartheta Be$, and ϑ can be calculated as the

³Note that our model needs to be lagged by one period in order to match the notation (and code) of Sims.

⁴A unitary matrix Q satisfies $Q'Q = QQ' = I$. If Q and/or Z contain complex values, the transpositions reflect complex conjugation, that is, each complex entry is replaced by its conjugate and then transposed.

ratio of diagonal elements of A and B , $\vartheta_i = \frac{\lambda_{ii}}{\omega_{ii}}$.⁵

Using these factorization, pre-multiply both sides of the equation by Q . The original system can be then expressed as

$$\Lambda z_t = \Omega z_{t-1} + QCv_t + QD\eta_t + QE, \quad (31)$$

where $z_{t+1} = Z'x_{t+1}$.⁶

Step 2:

Partition equation (31) into explosive and non-explosive blocks:

$$\begin{bmatrix} \Lambda_{11} & \Lambda_{12} \\ 0 & \Lambda_{22} \end{bmatrix} \begin{bmatrix} z_{1t} \\ z_{2t} \end{bmatrix} = \begin{bmatrix} \Omega_{11} & \Omega_{12} \\ 0 & \Omega_{22} \end{bmatrix} \begin{bmatrix} z_{1t-1} \\ z_{2t-1} \end{bmatrix} + \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} [Cv_t + D\eta_t + E]. \quad (32)$$

The model can now be solved iteratively.

Step 3:

To solve the lower (explosive) part of the system, z_{2t} , note that the lower block of (32) can be written as

$$\Lambda_{22}z_{2t} = \Omega_{22}z_{2t-1} + w_{2t}, \quad (33)$$

where $w_t = Q[Cv_t + D\eta_t + E]$ and w_{1t} and w_{2t} are partitioned conformably. Using equation (33) we can compute z_{2t} as

$$z_{2t} = M_{22}z_{2t+1} - \Omega_{22}^{-1}w_{2t+1}, \quad (34)$$

where $M = \Omega_{22}^{-1}\Lambda_{22}$. Iterating forward yields the expression for z_{2t}

$$z_{2t} = - \sum_{i=0}^{\infty} M^i \Omega_{22}^{-1} w_{2t+1+i}, \quad (35)$$

since $\lim_{t \rightarrow \infty} M^t z_{2t} = 0$. (Recall that $M = \Omega_{22}^{-1}\Lambda_{22}$ and generalized eigenvalues expressed as diagonal elements λ_{ii}/ω_{ii} are all outside unit circle in the lower block of the

⁵Sims' website provides a program that computes and orders the eigenvalues appropriately.

⁶Since Q is unitary $QQ' = I$.

system.)

Since w_t is defined as $w_t = Q[Cv_t + D\eta_t + E]$, equation (35) expresses z_{2t} as a function of future values of structural and expectational errors.

Since z_{2t} is known at time t , and $E_t(\eta_{t+s}) = E_t(v_{t+s}) = 0$ for $s > 0$ equation (35) can be written as

$$z_{2t} = - \sum_{i=0}^{\infty} M^i \Omega_{22}^{-1} Q_2 E_2, \quad (36)$$

where $Q_2 E_2$ are the lower portion of QE conformable with z_2 .⁷

Since $-\sum_{i=0}^{\infty} M^i = -(I - M)^{-1}$, the solution of z_{2t} is obtained as

$$z_{2t} = (\Lambda_{22} - \Omega_{22})^{-1} Q_2 E_2. \quad (37)$$

Step 4:

The final step is to solve for z_{1t} in equation (32). Note that the solution of z_{1t} requires a solution for the expectations errors that appear in (32).

Sims notes that if there is a unique solution for the model, there exist a systematic relationship exists between the expectations errors associated with z_{1t} and z_{2t} .

Necessary and sufficient condition for uniqueness

If there exists a $k \times (n - k)$ matrix Φ that satisfies

$$Q_1 D = \Phi Q_2 D, \quad (38)$$

the equilibrium is unique.

Φ represents the systematic relationship between the expectations errors associated with z_{1t} and z_{2t} noted above.

If there exists a unique equilibrium we can calculate Φ , as in equation (38), and the solution of z_{1t} can be computed. If we pre-multiply (31)

$$\begin{bmatrix} \Lambda_{11} & \Lambda_{12} \\ 0 & \Lambda_{22} \end{bmatrix} \begin{bmatrix} z_{1t} \\ z_{2t} \end{bmatrix} = \begin{bmatrix} \Omega_{11} & \Omega_{12} \\ 0 & \Omega_{22} \end{bmatrix} \begin{bmatrix} z_{1t-1} \\ z_{2t-1} \end{bmatrix} + \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} [Cv_t + D\eta_t + E].$$

⁷Sims also considers the case in which the structural innovations v_t are serially correlated, which leads to a generalization of (36).

by $[I - \Phi]$, we obtain

$$\begin{bmatrix} \Lambda_{11} & \Lambda_{12} - \Phi\Lambda_{22} \end{bmatrix} \begin{bmatrix} z_{1t} \\ z_{2t} \end{bmatrix} = \begin{bmatrix} \Omega_{11} & \Omega_{12} - \Phi\Omega_{22} \end{bmatrix} \begin{bmatrix} z_{1t-1} \\ z_{2t-1} \end{bmatrix} + \begin{bmatrix} Q_1 - \Phi Q_2 \end{bmatrix} [Cv_t + D\eta_t + E]. \quad (39)$$

If equation (38) holds, the loading factor for the expectational errors, η_t , is zero. Then the system may be written in the form

$$x_t = \Theta_e + \Theta_0 x_{t-1} + \Theta_1 v_t, \quad (40)$$

where

$$H = Z \begin{bmatrix} \Lambda_{11}^{-1} & -\Lambda_{11}^{-1}(\Lambda_{12} - \Phi\Lambda_{22}) \\ 0 & I \end{bmatrix} \quad (41)$$

$$\Theta_e = H \begin{bmatrix} Q_1 - \Phi Q_2 \\ (\Omega_{22} - \Lambda_{22})^{-1} Q_2 \end{bmatrix} E \quad (42)$$

$$\Theta_0 = Z \Lambda_{11}^{-1} [\Omega_{11} (\Omega_{12} - \Phi\Omega_{22})] Z' \quad (43)$$

$$\Theta_1 = H \begin{bmatrix} Q_1 - \Phi Q_2 \\ 0 \end{bmatrix} D \quad (44)$$