Lecture: Solving RBC model

Advanced Macroeconomics
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Plan of the Presentation

1. Linearization

2. Solution
   - Blanchard and Kahn's (1980) Method
   - Sims' (2001) Method

What we did

- We built the model
- We have a non-linear expectational difference model
- Want to solve it. We can either
  1. use non-linear solution method,
  2. use (linear) approximation method.
- Today:
  - approximate (linearly) using Taylor-series expansions and solve the resulting linear system.
- If we log-linearize the system, variables are expressed as deviations from steady state values.

Taylor series approximation

- Consider an arbitrary univariate function $f : R \rightarrow R$.
- Taylor's theorem says that if f is k-times differentiable then $f(x)$ can be approximated by a power series around a particular point x^*:

$$f(x) = f(x^*) + \frac{f'(x^*)}{1!}(x - x^*) + \frac{f''(x^*)}{2!}(x - x^*)^2 + \frac{f^{(3)}(x^*)}{3!}(x - x^*)^3 + \dots$$
$$+ \frac{f^{(k)}(x^*)}{k!}(x - x^*)^k + h_k(x)(x - x^*)^k$$

  - $f'(x^*)$ denotes the first derivative of f with respect to x, evaluated at x^*;
  - $f''(x^*)$ is the second derivative at x^*;
  - $f^{(n)}(x^*)$ is n-th derivative at x^*
  - $\lim_{x \rightarrow x^*} h_k(x) = 0$.
- The order of approximation tells how many powers are used to approximate
  - 1^{st} order (linear) approximation

$$f(x) \simeq f(x^*) + f'(x^*)(x - x^*)$$

  - 2^{nd} order (quadratic) approximation

$$f(x) \simeq f(x^*) + f'(x^*)(x - x^*) + \frac{f''(x^*)}{2}(x - x^*)^2$$

Taylor series approximation

- Taylor's theorem holds for multivariate functions.
- Consider a function $f(x) = f(x_1, x_2), x = [x_1 \quad x_2]'$ $k-$times differentiable at x^*.
- Taylor series around $x^* = (x_1^*, x_2^*)$ can be written as

$$f(x_1,x_2) = f(x^*) + \frac{\partial f}{\partial x_1}(x^*)(x_1 - x_1^*) + \frac{\partial f}{\partial x_2}(x^*)(x_2 - x_2^*) + \\ + \frac{1}{2}\frac{\partial^2 f}{\partial x_1^2}(x^*)(x_1 - x_1^*)^2 + \frac{1}{2}\frac{\partial^2 f}{\partial x_2^2}(x^*)(x_2 - x_2^*)^2 \\ + \frac{\partial^2 f}{\partial x_2 \partial x_1}(x^*)(x_1 - x_1^*)(x_2 - x_2^*) + \ldots$$

  - $\frac{\partial f}{\partial x_i}$ denotes the partial derivative of f with respect to x_i, evaluated at x^*;

Log-linearization

- Sometimes it is more convenient to first take log of these equations and linearize them.
- Consider an equation $f(x_t) = g(x_t)$. Taking logs of both sides yields

$$\ln f(x_t) = \ln g(x_t)$$

- Now, if we take first-order Taylor expansion around $x_t = x^*$ we get

$$\ln f(x_t) \approx \ln f(x^*) + \frac{f'(x^*)}{f(x^*)}(x_t - x^*)$$

$$\ln g(x_t) \approx \ln g(x^*) + \frac{g'(x^*)}{g(x^*)}(x_t - x^*)$$

$$\text{as} \quad [\ln f(x)]' = \frac{f'(x)}{f(x)}$$

- Substituting in

$$\ln f(x^*) + \frac{f'(x^*)}{f(x^*)}(x_t - x^*) = \ln g(x^*) + \frac{g'(x^*)}{g(x^*)}(x_t - x^*)$$

and using $\ln f(x^*) = \ln g(x^*)$ and rearranging terms we get

$$\frac{f'(x^*)x^*}{f(x^*)} \frac{x_t - x^*}{x^*} = \frac{g'(x^*)x^*}{g(x^*)} \frac{x_t - x^*}{x^*}$$

Log-linearization

- Define $\hat{x}_t = \frac{x_t - x^*}{x^*}$, or the percentage deviation of x_t from x^* to get

$$\frac{f'(x^*)x^*}{f(x^*)}\hat{x}_t = \frac{g'(x^*)x^*}{g(x^*)}\hat{x}_t$$

  - Note that $\frac{f'(x^*)x^*}{f(x^*)}$ is the elasticity of f with respect to x_t at x^*.

- Since $\ln(1+\epsilon) \approx \epsilon$ for small values of ϵ, we can write $\hat{x}_t$ as

$$\hat{x}_t \approx \ln(\hat{x}_t + 1) = \ln\left(\frac{x_t - x^*}{x^*} + 1\right) = \ln\left(\frac{x_t}{x^*}\right) = \ln x_t - \ln x^*$$

and, as $x = e^{\ln x} = \ln e^x$,

$$x_t = x^* \frac{x_t}{x^*} = x^* e^{\ln\left(\frac{x_t}{x^*}\right)} \approx x^* e^{\hat{x}_t}.$$

- Lastly, note that the Taylor expansions of exponential function and the product of two variables around $\hat{x}_t = \hat{y}_t = 0$ are

$$\exp(\alpha\hat{x}_t) \approx \exp(\alpha\hat{x}_t)|_{\hat{x}_t=0} + \alpha\exp(\alpha\hat{x}_t)|_{\hat{x}_t=0}\hat{x}_t = 1 + \alpha\hat{x}_t$$
$$\hat{x}_t\hat{y}_t \approx 0 + \hat{y}_t|_{\hat{y}_t=0}(\hat{x}_t - 0) + \hat{x}_t|_{\hat{x}_t=0}(\hat{y}_t - 0) = 0$$

Log-linearization

- Using these observations we obtain several useful expressions

$$
\begin{aligned}
x_t &= x^* e^{\hat{x}_t} \approx x^* \cdot (1 + \hat{x}_t) \\
x_t y_t &= x^* e^{\hat{x}_t} y^* e^{\hat{y}_t} \approx x^* y^* \cdot (1 + \hat{x}_t + \hat{y}_t) \\
x_t^a &\approx (x^*)^a \cdot (1 + a\hat{x}_t) \\
x_t^a y_t^b &\approx (x^*)^a (y^*)^b \cdot (1 + a\hat{x}_t + b\hat{y}_t)
\end{aligned}
$$

Back to model - summary

- Recall we have a system of 8 equations with 8 endogenous variables (c, r, l, w, k, i, y, z):

$$c_t^{-\sigma} = \beta E_t[c_{t+1}^{-\sigma}(1 + r_{t+1} - \delta)] \tag{1}$$

$$w_t = \frac{l_t^{\varphi}}{c_t^{-\sigma}} \tag{2}$$

$$k_t = (1 - \delta)k_{t-1} + i_t \tag{3}$$

$$y_t = c_t + i_t \tag{4}$$

Back to model - summary

$$y_t = z_t k_{t-1}^{\alpha} l_t^{1-\alpha} \tag{5}$$

$$w_t = (1-\alpha)\frac{y_t}{l_t} \tag{6}$$

$$r_t = \alpha \frac{y_t}{k_{t-1}} \tag{7}$$

$$z_t = exp(\epsilon_t) z_{t-1}^{\rho} \tag{8}$$

- We computed steady-states last time.
- Now we will linearize each equation.

Log-linearize labor - consumption choice

Equation (2)

$$\frac{l_t^{\varphi}}{c_t^{-\sigma}} = w_t$$

In the steady state:

$$\frac{(l^{ss})^{\varphi}}{(c_t^{ss})^{-\sigma}} = w^{ss}$$

Let's log-linearize:

$$\frac{(l^{ss})^{\varphi}}{(c^{ss})^{-\sigma}}(1 + \varphi\hat{l}_t + \sigma\hat{c}_t) = w^{ss}(1 + \hat{w}_t)$$

Divide by steady state:

$$\varphi\hat{l}_t + \sigma\hat{c}_t = \hat{w}_t$$

Log-linearized Euler equation

Equation (1):

$$c_t^{-\sigma} = \beta E_t[c_{t+1}^{-\sigma}(1 + r_{t+1} - \delta)]$$

In the steady state:

$$1 = \beta(1 + r^{ss} - \delta)$$

When linearized:

$$\left(c^{ss}\right)^{-\sigma}(1 - \sigma\hat{c}_t) = \beta\left(c^{ss}\right)^{-\sigma} E_t\left[(1 - \sigma\hat{c}_{t+1})(1 - \delta + r^{ss}(1 + \hat{r}_{t+1}))\right]$$

Log-linearized Euler equation

Substitute for r^{ss}:

$$1 - \sigma\hat{c}_t = \beta E_t \left[(1 - \sigma\hat{c}_{t+1})(1 - \delta + (\frac{1}{\beta} - 1 + \delta)(1 + \hat{r}_{t+1})) \right]$$

$$1 - \sigma\hat{c}_t = E_t \left[(1 - \sigma\hat{c}_{t+1})(\beta - \beta\delta + (1 - \beta + \beta\delta)(1 + \hat{r}_{t+1})) \right]$$

$$1 - \sigma\hat{c}_t = E_t \left[(1 - \sigma\hat{c}_{t+1})(1 + (1 - \beta(1 - \delta))\hat{r}_{t+1}) \right]$$

Multiply and drop higher order terms:

$$1 - \sigma\hat{c}_t = 1 - \sigma E_t\hat{c}_{t+1} + (1 - \beta(1 - \delta))E_t\hat{r}_{t+1}$$

Rearrange terms:

$$\sigma(E_t\hat{c}_{t+1} - \hat{c}_t) = (1 - \beta(1 - \delta))E_t\hat{r}_{t+1}$$

Log-linearized market clearing condition

Equation (4):

$$c_t + i_t = y_t$$

In the steady state:

$$c^{ss} + i^{ss} = y^{ss}$$

Linearize:

$$c^{ss}(1 + \hat{c}_t) + i^{ss}(1 + \hat{i}_t) = y^{ss}(1 + \hat{y}_t)$$

Subtract steady state equation:

$$(1 - \frac{i^{ss}}{y^{ss}})\hat{c}_t + \frac{i^{ss}}{y^{ss}}\hat{i}_t = \hat{y}_t$$

Log-linearized firm's equilibrium conditions

- For labor (equation 6):

$$(1-\alpha)\frac{y_t}{l_t} = w_t$$

Linearized:

$$\hat{y}_t - \hat{l}_t = \hat{w}_t$$

- For capital (equation 7):

$$\alpha\frac{y_t}{k_{t-1}} = r_t$$

Linearized:

$$\hat{y}_t - \hat{k}_{t-1} = \hat{r}_t$$

Log-linearized shock process

Equation (8):

$$z_t = exp(\epsilon_t) z_{t-1}^{\rho}$$

After linearization:

$$\hat{z}_t = \rho \hat{z}_{t-1} + \epsilon_t$$

Steady state values

- Our equations contain steady state ratio $\frac{i^{ss}}{y^{ss}}$.
- These are determined by our parameters

From the Euler equation:

$$r^{ss} = \beta^{-1} - (1 - \delta)$$

and from the equilibrium condition for capital:

$$\begin{aligned} r^{ss} k^{ss} &= \alpha y^{ss} \\ \frac{k^{ss}}{y^{ss}} &= \frac{\alpha}{r^{ss}} = \frac{\alpha}{\beta^{-1} - (1 - \delta)} \end{aligned}$$

Steady state values cont'd

From the capital accumulation equation:

$$\delta k^{ss} = i^{ss}$$

thus

$$\frac{i^{ss}}{y^{ss}} = \delta \frac{k^{ss}}{y^{ss}} = \frac{\alpha\delta}{\beta^{-1} - (1-\delta)}$$

Log-linearized system

- We now have a system of 8 linear (difference) equations and 8 variables
- Have to solve it

Plan of the Presentation

1. Linearization
2. Solution

Solving linear DSGE models

- Solving a DSGE model means changing the system of forward-looking difference equations that we have ...

  ... into a VAR , a system of backward-looking difference equations
- There are several techniques for solving such systems:

  e.g. Blanchard and Kahn (1980), Sims (2001), Christiano (2002), Uhlig (2002), Evans and Honkapohja (2001),...
- (Dynare will solve it for you)

Blanchard & Kahn condition

- One very important issue is the existence and the stability condition
- Write the system in state space form:

$$A_1 \begin{bmatrix} X_{t+1} \\ E_t P_{t+1} \end{bmatrix} = A_0 \begin{bmatrix} X_t \\ P_t \end{bmatrix} + \gamma Z_{t+1}$$

where:

- X_t: vector $n \times 1$ of state variables (backward-looking)
- P_t: vector $m \times 1$ of jumpers (forward-looking)
- Z_t: vector $k \times 1$ of shocks (with mean equal to 0 every period)
- A_1, A_0: $(n+m) \times (n+m)$ matrices
- γ: $(n+m) \times k$ matrix

Blanchard & Kahn condition cont'd

- Assume that A_1 is invertible. Then:

$$\left[\begin{array}{c} X_{t+1} \\ E_t P_{t+1} \end{array}\right] = A \left[\begin{array}{c} X_t \\ P_t \end{array}\right] + A_1^{-1} \gamma Z_{t+1}$$

where $A = A_1^{-1} A_0$

- The Blanchard-Kahn condition says that for the model to have a unique solution, the number of eigenvalues of A lying outside the unit circle (unstable roots) must equal the number of forward looking variables (jumpers)

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