Lecture: Some Informal Notes on Dynamic Programming

The purpose of these class notes is to give an informal introduction to dynamic programming by working out some cases "by hand."

We want to solve

$$\max_{\{c_t,k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t U(c_t)$$

subject to

$$c_t = f(k_t) + (1-\delta)k_t - k_{t+1}$$

with k_0 given. Equivalently, the problem that we want to address is the selection of a sequence $\{k_{t+1}\}_{t=0}^{\infty}$ that solves

$$\max_{\{k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t U(f(k_t) + (1-\delta)k_t - k_{t+1})$$

with k_0 given. Here f is a production function assumed to satisfy the standard Inada conditions. Note that the sum is converging as $0 < \beta < 1$ and k is bounded at every t.

1 Finite Horizon Problem

To begin, assume that the time horizon is finite and equal to T, so that the problem is to find a sequence $\{k_{t+1}\}_{t=0}^{T}$ to solve

$$\max_{\{k_{t+1}\}_{t=0}^{T}} \sum_{t=0}^{T} \beta^t U(f(k_t) + (1-\delta)k_t - k_{t+1})$$

with k_0 given. A simple way to address this problem is by backward induction.

Time T: To keep notation simple, let $F(k_t) \equiv f(k_t) + (1-\delta)k_t$, and assume to be at time T. Then, $k_0, \ldots, k_T$ are given by history, and the maximization problem to solve the last period problem is simple:

$$\max_{k_{T+1}} \beta^T U(F(k_T) - k_{T+1}).$$

The solution to this problem sets $k_{T+1} = 0$ because at time $T+1$ there is no utility from consumption, so there is no point in accumulating capital. Define $V_0(k_T) = U[F(k_T)]$; this expression gives the maximum attainable utility at time T, given the capital level k_T inherited historically and the optimal choice of k_{T+1}. That is, V_0 represents the *value* of solving the problem at T, when there are zero periods to go and the capital stock available, k_T, is given.

Time T − 1: Consider the situation at $T-1$. Now, $k_0, \dots, k_{T-1}$ are historically given. The problem can be written as

$$\max_{k_T} \beta^{T-1} U(F(k_{T-1}) - k_T) + \beta^T V_0(k_T).$$

The first-order condition is given by

$$U'(F(k_{T-1}) - k_T) = \beta V_0'(k_T), \qquad \text{(FOC } T-1\text{)}$$

and regularity of U and F implies that the first-order condition has a solution given by[1]

$$k_T = g_T(k_{T-1}).$$

Clearly, the solution will be a function of k_{T-1}, and we want to know something about the map g_T. Note that differentiating both sides of equation (FOC $T-1$) with respect to k_{T-1} yields

$$U''(\cdot)\left(F'(\cdot) - \frac{dk_T}{dk_{T-1}}\right) = \beta V_0''(\cdot)\frac{dk_T}{dk_{T-1}},$$

which implies

$$\frac{dk_T}{dk_{T-1}} = g_T'(k_{t-1}) = \left(\frac{U''}{U'' + \beta V_0''}\right) F' > 0.$$

Therefore, a higher value of k_{T-1} leads to a higher choice of k_T. In addition, $g_T'(k_{T-1}) < F'(k_{T-1})$ and, because $c_{T-1} = F(k_{T-1}) - k_T$, we have

$$\frac{dc_{T-1}}{dk_{T-1}} = F' - g_T' > 0.$$

[1] Note that $V_0' = U'F' > 0$ and $V_0'' = U''F' + U'F'' < 0$.

Define

$$V_1(k_{T-1}) = U[F(k_{T-1}) - g_T(k_{T-1})] + \beta V_0(g_T(k_{T-1})),$$

where from (FOC $T-1$)

$$V_1' = U'(F' - g_T') + \beta V_0' g_T' = U'F' > 0$$

and, since $U'', F'' < 0$, $U', F' > 0$, and $(F' - g_T') > 0$,

$$V_1'' = U''(F' - g_T')F' + U'F'' < 0,$$

and $k_T = g_T(k_{T=1})$.

Time T − 2: At $t = T-2$, $k_0, \ldots, k_{T-2}$ are given, and the problem is to choose k_{T-1}, k_T, and k_{T+1}. But we already know that $k_{T+1} = 0$ and $k_T = g_T(k_{T-1})$. Then the problem is

$$\max_{k_{T-1}} \beta^{T-2} \left(U[F(k_{T-2}) - k_{T-1}] + \beta U[F(k_{T-1}) - g_T(k_{T-1})] + \beta^2 V_0[g_T(k_{T-1})] \right)$$
$$= \max_{k_{T-1}} U[F(k_{T-2}) - k_{T-1}] + \beta V_1[k_{T-1}].$$

The first-order condition for this problem is

$$U'(F(k_{T-2}) - k_{T-1}) = \beta V_1'(k_{T-1}),$$

which gives a solution

$$k_{T-1} = g_{T-1}(k_{T-2}).$$

In addition

$$U'' \left(F' - g_{T-1}' \right) = \beta V_1'' g_{T-1}',$$

or

$$g_{T-1}' = \frac{U''}{U'' + \beta V_1''} F',$$

with $0 < g_{T-1}' < F'$ as before.

Define

$$V_2(k_{T-2}) = U[F(k_{T-2}) - g_{T-1}(k_{T-2})] + \beta V_1[g_{T-1}(k_{T-2})].$$

with

$$\begin{aligned}
V_2' &= U'[F' - g_{T-1}'] + \beta V_1' g_{T-1}' \\
&= U'F' + g_{T-1}'[\beta V_1' - U'] \\
&= U'F' > 0
\end{aligned}$$

and

$$V_2'' = F''U' + F'U''(F' - g_{T-1}') < 0.$$

Note that the value function always summarizes what will happen in the future and that more or less the same thing is happening in every period. That is, the problem displays a recursive structure. The arguments are not the same every period but the structure of the problem is. We want a sequence $V_0(k_T), V_1(k_{T-1}), \ldots, V_T(k_0)$ with associated optimal choices $g_T(k_{T-1}) = k_T$, $g_{T-1}(k_{T-2}) = k_{T-1}$, …, $g_1(k_0) = k_1$.

In general,

$$V_n(k_{T-n}) = U[F(k_{T-n}) - k_{T-n+1}] + \beta V_{n-1}(k_{T-n+1})$$

and

$$\begin{aligned}
V_T(k_0) &= \max_{\{k_{t+1}\}_{t=0}^{T}} \sum_{t=0}^{T} \beta^t U[F(k_t) - k_{t+1}] \\
V_T(k_0) &= \max_{k_1} \{U[F(k_0) - k_1] + \beta V_{T-1}(k_1)\}.
\end{aligned}$$

2 Infinite Horizon Problem

Suppose now that the time horizon of the problem is infinite. Consider V_n. At any time n there is an infinite number of periods to go. The issue is whether there is a time invariant value function V that is the limit of the finite horizon case. For the problems that we will consider the answer to this question is “yes”.

Associated with V there is a time-invariant policy function g such that $k_{t+1} = g(k_t)$. More formally,

$$V_T(k_0) = \max_{\{k_{t+1}\}_{t=0}^{T}} \sum_{t=0}^{T} \beta^t U[F(k_t) - k_{t+1}]$$

and

$$\lim_{T\to\infty} V_T = V.$$

This is the idea behind Bellman’s principle of optimality with the general idea being that if you consider a sequence of maximal payoffs and remove the first n elements, the remaining “tail” of the sequence must still be optimal. Formally we can express this idea as follows:

$$\begin{aligned} V(k_0) &= \max_{\{k_{t+1}\}_{t=0}^{T}} \sum_{t=0}^{\infty} \beta^t U[F(k_t) - k_{t+1}] \\ &= \max_{\{k_{t+1}\}_{t=0}^{T}} \left\{ U[F(k_0) - k_1] + \beta \sum_{t=1}^{\infty} \beta^{t-1} U[F(k_t) - k_{t+1}] \right\} \\ &= \max_{k_1} \left\{ U[F(k_0) - k_1] + \beta V(k_1) \right\}. \end{aligned}$$

Maximization of the infinite stream of payoffs is equivalent to maximizing today’s payoff followed by the maximal continuation value. The equation

$$V(k_0) = \max_{k_1} \left\{ U[F(k_0) - k_1] + \beta V(k_1) \right\}$$

is an example of a functional equation and it is referred to as the Bellman equation. The unknown in the Bellman equation is the function V.[2]

For a special class of payoff functions and transition equations the Bellman equation can be solved explicitly “by hand”. These case include

$$U(c_t) = \ln c_t$$

[2]We will mention the problem of proving the existence, uniqueness, and differentiability of V.

and

$$F(k_t) = Ak^{\alpha},$$

with $\delta = 1$. In this case there are two easy methods that can be applied to obtain the value function: (i) the method of undetermined coefficients, and (ii) the value function iteration.

2.1 Undetermined coefficients

For the method of undetermined coefficients we guess that

$$V(k_t) = B + b \ln k_t.$$

We then verify that this is an admissible solution of the Bellman equation. To do so we solve:

$$\max_{\{c_t, k_{t+1}\}} U(c_t) + \beta V(k_{t+1})$$

or, substituting $c_t = Ak_t^{\alpha} - k_{t+1}$ and our guess for $V(k)$,

$$\max_{k_{t+1}} \ln[Ak_t^{\alpha} - k_{t+1}] + \beta[B + b \ln k_{t+1}]$$

taking k_t as given.

The first-order condition is

$$\begin{aligned} \frac{1}{Ak_t^{\alpha} - k_{t+1}} &= \frac{\beta b}{k_{t+1}} \\ \Longrightarrow \qquad k_{t+1} &= \frac{\beta b A k_t^{\alpha}}{1 + \beta b} \end{aligned}$$

Plugging the first-order condition into

$$V(k_t) = \max_{k_{t+1}} \{U(c_t) + \beta V(k_{t+1})\}$$

and using our guess for $V(k_t)$, we know that

$$V(k_t) = \ln\left(Ak_t^\alpha - \frac{\beta b A k_t^\alpha}{1+\beta b}\right) + \beta B + \beta b \ln \frac{\beta b A k_t^\alpha}{1+\beta b}.$$

Our guess for V implies that

$$V(k_t) = B + b\ln k_t = \ln\left(Ak_t^\alpha - \frac{\beta b A k_t^\alpha}{1+\beta b}\right) + \beta B + \beta b \ln \frac{\beta b A k_t^\alpha}{1+\beta b}$$

and it holds for all k_t.

We choose B and b so that our guess is verified. Expanding the right-hand side of this equation we rewrite it as:

$$\begin{aligned} V(k_t) &= \alpha \ln k_t + \ln\left(A - \frac{\beta b A}{1+\beta b}\right) + \beta B + \beta b \alpha \ln k_t + \beta b \ln\left(\frac{\beta b A}{1+\beta b}\right) \\ &= \ln\left(A - \frac{\beta b A}{1+\beta b}\right) + \beta B + \beta b \ln\left(\frac{\beta b A}{1+\beta b}\right) + \alpha(1+\beta b)\ln k_t. \end{aligned}$$

Since this must be equal to our guess $V(k_t) = B + b\ln k_t$, collecting the terms yield the following two equations:

$$\alpha(1+\beta b) = b,$$

which equates coefficient associated with $\ln k_t$ in both equations, and

$$\ln\left(A - \frac{\beta b A}{1+\beta b}\right) + \beta B + \beta b \ln\left(\frac{\beta b A}{1+\beta b}\right) = B,$$

which equate the remaining terms.

Solving the first equation for b yields

$$b = \frac{\alpha}{1-\alpha\beta}.$$

The second equation can be rewritten as

$$(1-\beta)B = \ln\left(\frac{A(1+\beta b)-\beta bA}{1+\beta b}\right) + \beta b \ln\left(\frac{\beta bA}{1+\beta b}\right)$$
$$(1-\beta)B = \ln A - \ln(1+\beta b) + \beta b \ln(\beta bA) - \beta b \ln(1+\beta b)$$
$$(1-\beta)B = (1+\beta b)\ln A - (1+\beta b)\ln(1+\beta b) + \beta b \ln(\beta b).$$

Substituting the expression for b yields

$$(1-\beta)B = \frac{1}{1-\alpha\beta}\left(\ln A - \ln\left(\frac{1}{1-\alpha\beta}\right) + \alpha\beta \ln\frac{\alpha\beta}{1-\alpha\beta}\right)$$
$$(1-\beta)B = \frac{1}{1-\alpha\beta}\left(\ln A - (1-\alpha\beta)\ln\left(\frac{1}{1-\alpha\beta}\right) + \alpha\beta\ln(\alpha\beta)\right)$$
$$(1-\beta)B = \frac{1}{1-\alpha\beta}\ln A + \ln(1-\alpha\beta) + \frac{\alpha\beta}{1-\alpha\beta}\ln(\alpha\beta)$$

and finally, after adding and subtracting $\frac{\alpha\beta}{1-\alpha\beta}\ln A$ from the right-hand side, we arrive at the expression for B:

$$B = \frac{1}{1-\beta}\left[\ln(A[1-\alpha\beta]) + \frac{\alpha\beta}{1-\alpha\beta}\ln(A\alpha\beta)\right].$$

We have shown, therefore, that the value function is given by

$$V(k_t) = B + b\ln k_t.$$

Substituting the value for b in the policy function we get

$$k_{t+1} = \beta\alpha A k_t^{\alpha}.$$

This is the same result that we get when we solve the maximization problem with the Euler equation approach. To see that, consider the problem

$$\max_{\{k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \ln(Ak_t^{\alpha} - k_{t+1}).$$

The first order condition at time t, the Euler equation, is

$$\frac{\beta^t}{Ak_t^\alpha - k_{t+1}} = \frac{\beta^{t+1}}{Ak_{t+1}^\alpha - k_{t+2}} \alpha A k_{t+1}^{\alpha-1}.$$

Rearranging the terms yields

$$Ak_{t+1}^\alpha - k_{t+2} = \beta \left(Ak_t^\alpha - k_{t+1}\right) \alpha A k_{t+1}^{\alpha-1}.$$

Since

$$c_t = Ak_t^\alpha - k_{t+1}$$

we get

$$c_{t+1} = \beta c_t \left(\alpha A k_{t+1}^{\alpha-1}\right).$$

Given the resource constraint

$$k_{t+1} + c_t = Ak_t^\alpha,$$

we have the system of two first order difference equations that gives the solution to our problem.

From the solution to the Bellman equation (remember that the solution to the Bellman equation is the value function as the solution to a functional equation is a function) we know that $k_{t+1} = \beta\alpha A k_t^\alpha$. The resource constraint then tells us that

$$c_t = (1 - \alpha\beta) A k_t^\alpha.$$

Shifting it by one period

$$c_{t+1} = (1 - \alpha\beta) A k_{t+1}^\alpha$$

and plugging it into the first-order condition yields

$$(1 - \alpha\beta) A k_{t+1}^\alpha = \beta c_t \alpha A k_{t+1}^{\alpha-1},$$

or

$$(1 - \alpha\beta) k_{t+1} = \alpha\beta c_t.$$

Using the resource constraint $c_t = Ak_t^\alpha - k_{t+1}$, we obtain

$$k_{t+1} = \beta\alpha A k_t^\alpha,$$

which is policy function we obtained with the Bellman equation.

2.2 Value Function Iteration

An important result in dynamic programming is that the sequence of value functions that solves the finite horizon problem converges (uniformly under some conditions) to the value function that solves the infinite horizon problem. A consequence of this result is that another way to solve dynamic programming and finding the policy function is by iterating the value function. We will apply this method to the problem that we have addressed so far, and show that we obtain the same solution.

The problem is, once again,

$$\max_{\{c_t, k_{t+1}\}_{t=0}^\infty} \sum_{t=0}^\infty \beta^t U(c_t)$$

subject to

$$c_t = Ak_t^\alpha - k_{t+1}$$

with k_0 given.

An easy way to start the iteration process is to choose $V_0(k_T) = 0$, where the meaning of V_0 is the same as before—there are no periods left. Since at the end of time the value of future capital is zero this guess is reasonable. With this initial choice, working backwards as in the finite horizon case, we know:

$$V_1(k_{T-1}) = \max_{k_T} \ln(Ak_{T-1}^\alpha - k_T) + \beta V_0(k_T)$$

which has an easy solution given that $V_0(k_T) = 0$: $k_T = 0$. With this solution we have that $c_{T-1} = Ak_{T-1}^\alpha$ and, therefore,

$$V_1(k_{T-1}) = \ln A + \alpha \ln k_{T-1}.$$

The next problem is then to determine V_2. The Bellman equation is

$$V_2(k_{T-2}) = \max_{k_{T-1}} \ln(Ak_{T-2}^{\alpha} - k_{T-1}) + \beta V_1(k_{T-1})$$

that is

$$V_2(k_{T-2}) = \max_{k_{T-1}} \ln(Ak_{T-2}^{\alpha} - k_{T-1}) + \beta \left(\ln A + \alpha \ln k_{T-1}\right).$$

The first-order condition for this problem is

$$\frac{1}{Ak_{T-2}^{\alpha} - k_{T-1}} = \frac{\alpha\beta}{k_{T-1}}$$

which implies

$$k_{T-1} = \left(\frac{\alpha\beta}{1+\alpha\beta}\right) Ak_{T-2}^{\alpha}$$

and

$$c_{T-2} = Ak_{T-2}^{\alpha} - k_{T-1} = \left(\frac{1}{1+\alpha\beta}\right) Ak_{T-2}^{\alpha}.$$

The value function V_2 becomes

$$\begin{aligned} V_2(k_{T-2}) &= \ln\left(\frac{1}{1+\alpha\beta} Ak_{T-2}^{\alpha}\right) + \beta\left[\ln A + \alpha \ln\left(\frac{\alpha\beta}{1+\alpha\beta} Ak_{T-2}^{\alpha}\right)\right] \\ &= \ln\left(\frac{A}{1+\alpha\beta}\right) + \beta \ln A + \alpha\beta \ln\left(\frac{\alpha\beta A}{1+\alpha\beta}\right) + \alpha(1+\alpha\beta)\ln k_{T-2}. \end{aligned}$$

Now let

$$\begin{aligned} v_0^2 &= \ln\left(\frac{A}{1+\alpha\beta}\right) + \beta \ln A + \alpha\beta \ln\left(\frac{\alpha\beta A}{1+\alpha\beta}\right) \\ v_1^2 &= \alpha(1+\alpha\beta) \end{aligned}$$

so the $V_2(k_{T-2})$ can be expressed as

$$V_2(k_{T-2}) = v_0^2 + v_1^2 \ln k_{T-2}.$$

Moving another step backward,

$$V_3(k_{T-3}) = \max_{k_{T-2}} \ln c_{T-3} + \beta V_2(k_{T-2})$$

or equivalently,

$$V_3(k_{T-3}) = \max_{k_{T-2}} \ln(Ak_{T-3}^{\alpha} - k_{T-2}) + \beta\left(v_0^2 + v_1^2 \ln k_{T-2}\right).$$

The first-order condition for this problem is given by

$$\frac{1}{Ak_{T-3}^{\alpha} - k_{T-2}} = \frac{\alpha\beta(1+\alpha\beta)}{k_{T-2}}$$

or

$$k_{T-2} = (Ak_{T-3}^{\alpha} - k_{T-2})\alpha\beta(1+\alpha\beta).$$

This yields

$$k_{T-2} = \left[\frac{1}{1+\alpha\beta+(\alpha\beta)^2}\right] Ak_{T-3}^{\alpha}$$

and

$$c_{T-3} = \left[\frac{\alpha\beta+(\alpha\beta)^2}{1+\alpha\beta+(\alpha\beta)^2}\right] Ak_{T-3}^{\alpha}.$$

Given this expressions, the value function can be written as

$$\begin{aligned} V_3(k_{T-3}) = &\ln\left[\frac{\alpha\beta+(\alpha\beta)^2}{1+\alpha\beta+(\alpha\beta)^2} Ak_{T-3}^{\alpha}\right] \\ &+ \beta\ln\left(\frac{A}{1+\alpha\beta}\right) + \beta^2 \ln A + \alpha\beta^2 \ln\left(\frac{\alpha\beta A}{1+\alpha\beta}\right) + \alpha\beta(1+\alpha\beta)\ln\left[\frac{1}{1+\alpha\beta+(\alpha\beta)^2} Ak_{T-3}^{\alpha}\right]. \end{aligned}$$

This can be simplified to

$$\begin{aligned} V_3(k_{T-3}) = &\ln\left[\frac{\alpha\beta+(\alpha\beta)^2}{1+\alpha\beta+(\alpha\beta)^2} A\right] \\ &+ \beta\ln\left(\frac{A}{1+\alpha\beta}\right) + \beta^2 \ln A + \alpha\beta^2 \ln\left(\frac{\alpha\beta A}{1+\alpha\beta}\right) \\ &+ \alpha\beta(1+\alpha\beta)\ln\left[\frac{A}{1+\alpha\beta+(\alpha\beta)^2}\right] + \alpha(1+\alpha\beta+(\alpha\beta)^2)\ln k_{T-3}. \end{aligned}$$

As before, define

$$
\begin{aligned}
v_0^3 =& \ln\left[\frac{\alpha\beta+(\alpha\beta)^2}{1+\alpha\beta+(\alpha\beta)^2}A\right] + \beta\ln\left(\frac{A}{1+\alpha\beta}\right) + \beta^2\ln A \\
&+ \alpha\beta^2\ln\left(\frac{\alpha\beta A}{1+\alpha\beta}\right) + \alpha\beta(1+\alpha\beta)\ln\left[\frac{A}{1+\alpha\beta+(\alpha\beta)^2}\right] \\
v_1^3 =& \alpha(1+\alpha\beta+(\alpha\beta)^2)
\end{aligned}
$$

so that

$$V_3(k_{T-3}) = v_0^3 + v_1^3 \ln k_{T-3}.$$

We move another step backward to compute V_4. The problem we have to address is

$$V_4(k_{T-4}) = \max_{k_{T-3}} \ln c_{T-4} + \beta V_3(k_{T-3})$$

or, equivalently,

$$V_4(k_{T-4}) = \max_{k_{T-3}} \ln(Ak_{T-4}^{\alpha} - k_{T-3}) + \beta\left(v_0^3 + v_1^3 \ln k_{T-3}\right).$$

The first-order condition for this problem is given by

$$\frac{1}{Ak_{T-4}^{\alpha} - k_{T-3}} = \frac{\alpha\beta\left(1+\alpha\beta+(\alpha\beta)^2\right)}{k_{T-3}}$$

or

$$k_{T-2} = (Ak_{T-3}^{\alpha} - k_{T-2})\alpha\beta(1+\alpha\beta).$$

This yields

$$k_{T-3} = \left[\frac{1}{1+\alpha\beta+(\alpha\beta)^2+(\alpha\beta)^3}\right] Ak_{T-4}^{\alpha}$$

and

$$c_{T-3} = \left[\frac{\alpha\beta+(\alpha\beta)^2+(\alpha\beta)^3}{1+\alpha\beta+(\alpha\beta)^2+(\alpha\beta)^3}\right] Ak_{T-4}^{\alpha}.$$

We can then write

$$V_4(k_{T-4}) = \ln\left[\frac{\alpha\beta+(\alpha\beta)^2+(\alpha\beta)^3}{1+\alpha\beta+(\alpha\beta)^2+(\alpha\beta)^3}Ak_{T-4}^{\alpha}\right] + \beta V_3(k_{T-3}) + \beta\left(v_0^3 + v_1^3\ln k_{T-3}\right).$$

Writing it out yields

$$\begin{aligned}
V_4(k_{T-4}) =& \ln\left[\frac{\alpha\beta+(\alpha\beta)^2+(\alpha\beta)^3}{1+\alpha\beta+(\alpha\beta)^2+(\alpha\beta)^3}Ak_{T-4}^{\alpha}\right] + \beta\ln\left[\frac{\alpha\beta+(\alpha\beta)^2}{1+\alpha\beta+(\alpha\beta)^2}A\right] \\
&+ \beta^2\ln\left(\frac{A}{1+\alpha\beta}\right) + \beta^3\ln A + \alpha\beta^3\ln\left(\frac{\alpha\beta A}{1+\alpha\beta}\right) \\
&+ \alpha\beta^2(1+\alpha\beta)\ln\left(\frac{A}{1+\alpha\beta+(\alpha\beta)^2}\right) \\
&+ (\alpha\beta+(\alpha\beta)^2+(\alpha\beta)^3)\ln\left[\frac{1}{1+\alpha\beta+(\alpha\beta)^2+(\alpha\beta)^3}Ak_{T-4}^{\alpha}\right].
\end{aligned}$$

We can collect terms to write

$$\begin{aligned}
V_4(k_{T-4}) =& \ln\left[\frac{\alpha\beta+(\alpha\beta)^2+(\alpha\beta)^3}{1+\alpha\beta+(\alpha\beta)^2+(\alpha\beta)^3}A\right] + \beta\ln\left[\frac{\alpha\beta+(\alpha\beta)^2}{1+\alpha\beta+(\alpha\beta)^2}A\right] \\
&+ \beta^2\ln\left(\frac{A}{1+\alpha\beta}\right) + \beta^3\ln A + \alpha\beta^3\ln\left(\frac{\alpha\beta A}{1+\alpha\beta}\right) \\
&+ \alpha\beta^2(1+\alpha\beta)\ln\left(\frac{A}{1+\alpha\beta+(\alpha\beta)^2}\right) \\
&+ \left(\alpha\beta+(\alpha\beta)^2+(\alpha\beta)^3\right)\ln\left[\frac{1}{1+\alpha\beta+(\alpha\beta)^2+(\alpha\beta)^3}A\right] \\
&+ \alpha\left[\alpha\beta+(\alpha\beta)^2+(\alpha\beta)^3\right]k_{T-4}.
\end{aligned}$$

As before, define

$$\begin{aligned}
v_0^4 =& \ln\left[\frac{\alpha\beta+(\alpha\beta)^2+(\alpha\beta)^3}{1+\alpha\beta+(\alpha\beta)^2+(\alpha\beta)^3}A\right] + \beta\ln\left[\frac{\alpha\beta+(\alpha\beta)^2}{1+\alpha\beta+(\alpha\beta)^2}A\right] \\
&+ \beta^2\ln\left(\frac{A}{1+\alpha\beta}\right) + \beta^3\ln A + \alpha\beta^3\ln\left(\frac{\alpha\beta A}{1+\alpha\beta}\right) \\
&+ \alpha\beta^2(1+\alpha\beta)\ln\left(\frac{A}{1+\alpha\beta+(\alpha\beta)^2}\right) \\
&+ \left(\alpha\beta+(\alpha\beta)^2+(\alpha\beta)^3\right)\ln\left[\frac{1}{1+\alpha\beta+(\alpha\beta)^2+(\alpha\beta)^3}A\right], \\
v_1^4 =& \alpha\left[\alpha\beta+(\alpha\beta)^2+(\alpha\beta)^3\right]
\end{aligned}$$

so that

$$V_4(k_{T-4}) = v_0^4 + v_1^4\ln k_{T-4}.$$

Note that some patterns are beginning to emerge. In general, we can guess that when we are solving the problem t periods from the end of time we get

$$v_1^t = \alpha \sum_{i=1}^{t} (\alpha\beta)^i,$$

so for infinitely many periods,

$$v_1^\infty = \lim_{t\to\infty} \alpha \sum_{i=1}^{t} (\alpha\beta)^i = \frac{\alpha}{1-\alpha\beta}.$$

For the other series, we can see that v_0^t is composed by two “sub series”. Let

$$x_t^i = \ln\left[\frac{A}{1+\alpha\beta+(\alpha\beta)^2+\ldots+(\alpha\beta)^{i-t-1}}\right]$$

and

$$y_t^i = \alpha\beta\left[1+\alpha\beta+(\alpha\beta)^2+\ldots+(\alpha\beta)^{i-t-2}\right]\ln\left[\frac{1+\alpha\beta+(\alpha\beta)^2+\ldots+(\alpha\beta)^{i-t-2}}{1+\alpha\beta+(\alpha\beta)^2+\ldots+(\alpha\beta)^{i-t-1}}\alpha\beta A\right].$$

v_0^i can be written as

$$v_0^i = \sum_{t=0}^{i-1} \beta^t x_t^i + \alpha\beta \sum_{t=0}^{i-2} \beta^t y_t^i.$$

Since

$$\lim_{i\to\infty} \sum_{t=0}^{i-1} \beta^t x_t^i = \left(\frac{1}{1-\beta}\right)\ln\left[A(1-\alpha\beta)\right]$$

and

$$\lim_{i\to\infty} \sum_{t=0}^{i-2} \beta^t y_t^i = \left(\frac{1}{1-\beta}\right)\left(\frac{\alpha\beta}{1-\alpha\beta}\right)\ln\left(A\alpha\beta\right)$$

we get

$$\lim_{i\to\infty} v_0^i = \left(\frac{1}{1-\beta}\right)\left(\ln\left[A(1-\alpha\beta)\right] + \frac{\alpha\beta}{1-\alpha\beta}\right)\ln\left(A\alpha\beta\right)$$

and the value function is

$$V(k_t) = \left(\frac{1}{1-\beta}\right)\left(\ln\left[A(1-\alpha\beta)\right] + \frac{\alpha\beta}{1-\alpha\beta}\right)\ln\left(A\alpha\beta\right) + \frac{\alpha}{1-\alpha\beta}\ln k_t$$

as we had determined with the method of undetermined coefficients.